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ENGINEERING SERIES

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ENGINEERING MECHANICS

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GINN AND COMPANY

BOSTON · NEW YORK · CHICAGO · LONDON
ATLANTA · DALLAS · COLUMBUS · SAN FRANCISCO

PRINTED IN THE UNITED STATES OF AMERICA

429.9

The Athenæum Press
GINN AND COMPANY · PROPRIETORS · BOSTON · U.S.A.

PREFACE

This text has been designed primarily for students of engineering who have had the usual first course in calculus. The student's viewpoint and limited experience have been kept constantly in mind with the hope that the text will prove readable to the average college student.

A concise discussion of units is given. The graphical methods of statics have been confined to one chapter. The catenary is treated somewhat in detail and its mechanical and geometric properties are indicated. The theory of virtual work has been included on account of its importance in modern engineering problems. A simplified treatment of moment of inertia is given, including the use of equimomental points. Kinematics is treated mainly by analytic methods and includes the theory of moving axes for plane motion.

The energy equation is developed early and used throughout dynamics. The rectilinear motion of a particle under the action of a force which is a function of the distance, time, or velocity serves to introduce some elementary differential equations. The motion of a particle on a revolving curve is discussed.

Experience has shown that the fundamental principles of mechanics can only be made a vital part of the equipment of an engineering student by the solution of numerous problems. Many illustrative examples have been fully solved in the text, and more than the usual number of problems have been included. The answers are given for all the problems.

The authors wish to acknowledge their indebtedness to the works of Routh and Appell. They are under obligation to Professor J. I. Parcel for his assistance in preparing the chapter on graphical statics, and to Professors Hartig, Boehnlein, Siler, Herrick, and Herrmann for their many helpful suggestions and criticisms while teaching the text in mimeograph form.

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UNIVERSITY OF MINNESOTA

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ENGINEERING MECHANICS

MECHANICS

CHAPTER I

INTRODUCTION

Mechanics is the science of the action of forces upon bodies. A body may be at rest or in motion under the action of forces.

Statics is that part of mechanics which is concerned with bodies at rest.

Dynamics is that part of mechanics which is concerned with bodies in motion.

Kinematics is that part of mechanics which is concerned with the motion of bodies, without reference to the causes producing or influencing the motion.

1. Mechanics. Mechanics is founded on a few apparently simple concepts and principles which have been derived from experience and intuition.

The object of mechanics is the solution of two fundamental problems:

(1) Given the forces which act on a system of bodies, to determine the motion or equilibrium of the bodies.

(2) Given the motion or equilibrium of a system of bodies, to find the forces.

In some branches of mathematics, *numbers* are the only quantities used. In geometry the concept of *space* in addition to that of numbers is used. In kinematics another concept, *time*, is added; and in mechanics *force*, *inertia*, *mass*, etc. are also employed.

2. Vectors. A quantity that is completely characterized by its numerical value is called a *scalar* quantity. Volume, mass, weight, work, temperature, and time are scalars. Several quantities occur in mechanics, however, which cannot be defined without reference to direction. A quantity that has *direction and also a numerical value* is called a vector quantity or *vector*. Velocity, acceleration, displacement, and force are vectors.

A scalar quantity is merely a number and obeys all the laws of ordinary algebra. A vector quantity involves direction as well as magnitude and has an algebra peculiar to itself.

A vector is represented by an arrow or a directed line segment. The length of the line to any convenient scale represents the magnitude of the vector, and the direction of the line represents the direction of the vector. The vector AB may be conveniently designated as $\overline{AB}$.

FIG. 1

3. Addition of vectors. In general two vectors have different directions and therefore cannot be added as scalars are added. By the addition of two vectors is meant the following process:

Let $\overline{PQ}$ and $\overline{RS}$ be any two vectors. Make $\overline{AB}$ equal and parallel to $\overline{PQ}$, also $\overline{BC}$ equal and parallel to $\overline{RS}$; then $\overline{AC}$ represents the sum of $\overline{PQ}$ and $\overline{RS}$. This may be written in the form of a vector equation:

$$\overline{PQ} + \overline{RS} = \overline{AB} + \overline{BC} = \overline{AC}.$$

Similarly, $$\overline{AB} + \overline{BC} + \overline{CD} = \overline{AC} + \overline{CD} = \overline{AD}.$$

This construction, or geometric addition, can be extended to any number of vectors in a plane or in space.

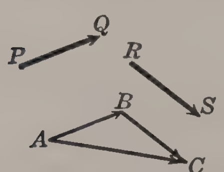

FIG. 2

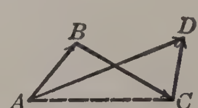

FIG. 3

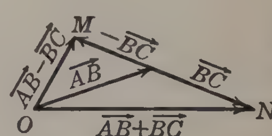

FIG. 4

The negative of any vector $\overline{BC}$ may be written $-\overline{BC}$ or $\overline{CB}$. Therefore to subtract a vector is to add its negative.

Hence $\overline{AB} + \overline{BC} = \overline{ON}$ and $\overline{AB} - \overline{BC} = \overline{OM}$.

4. Resolution of a vector. Any vector may be resolved into any number of other vectors by constructing a closed polygon of which the given vector is one side. Thus $\overline{AB}$ may be resolved into $\overline{AP} + \overline{PQ} + \overline{QR} + \overline{RB}$.

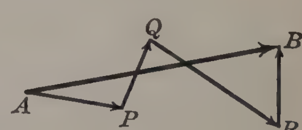

FIG. 5

Any vector may be specified by its length and its direction referred to some fixed line or axis. The notation of polar coördinates may be used.

5. Definitions. *Matter* is that which occupies space.

A *body* is a part of matter bounded by a closed surface.

A *particle* is a body indefinitely small in every direction. It may be considered as a geometric point endowed with the properties of matter.

A *rigid body* is a body whose particles do not change their distances from one another. Many bodies approximate the ideal rigid body.

Space and *time* are fundamental concepts. To our consciousness space extends in all directions and time flows on uniformly. All bodies are moving in space and no such thing as absolute rest is known. For most purposes in technical mechanics the earth is considered as a body at rest.

The units of length and time in the English system are the foot and second respectively.

The *position* of a body or particle means its relation in space to some other body or bodies taken as a standard of reference. When the position of a body is changing it is in *motion*.

6. Displacement. The *displacement* of a particle is its change of position. Thus, if a particle moves from A to B along any path whatever, the displacement is the distance AB measured along the straight line AB. The displacement of a particle from A to B followed by its displacement from B to C is equal to the displacement of the particle from A to C.

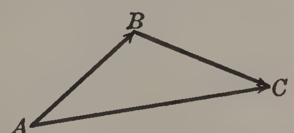

FIG. 6

Displacement is a directed quantity, that is, a vector. The addition of two displacements is accomplished by vectorial, or geometric, addition. Hence $\overline{AB} + \overline{BC} = \overline{AC}$.

7. Velocity and speed. When a particle is moving along a straight or curved path, the *speed* at any point P in the path is the limiting value of $\frac{\Delta s}{\Delta t}$ as Δt approaches zero, where Δs is the *length* of arc measured from the point P to a neighboring point P' on the path. The speed may be constant or variable; if equal lengths of arc are passed over in equal periods of time, for *any* and *every* choice of period, the speed is said to be constant or uniform.

The *velocity* of a particle at a point P is a vector whose magnitude is the speed of the particle at the point P and whose

direction is the direction of the tangent to the path of the particle at the point P. The sense of the tangent agrees with the sense of the motion of the point.

It is important to distinguish clearly between velocity and speed. Speed is a *scalar* quantity; velocity is a *vector*. If the *velocity* of a particle is uniform, its path must be a straight line; if its *speed* is uniform, its path may be any curve. Speed is usually expressed in feet per second. Velocity is also expressed in feet per second together with a definite direction.

8. Acceleration. When a particle moves along a path, its velocity may change from point to point. This change may be a change of speed or a change of direction or both. *Acceleration* is the measure of this change with respect to time. The unit acceleration is such that a unit velocity is acquired in unit time. In the British system of units the unit acceleration is usually taken as one foot per second per second. Acceleration always has a definite direction and is therefore a vector. (See § 124.)

9. Acceleration of gravity; weight. The acceleration of a freely falling body at any point of the earth's surface is called the *acceleration of gravity* and is denoted by g. Experiments show that it is the same for all bodies at the same place. It varies slightly with the location on the earth's surface.

Although the weight of a body has been given various meanings, it is here defined as follows: The *weight* W of a body at any place on the earth's surface is the pull or attraction which the earth exerts upon the body. Experiments show that the weight of a body varies slightly with its location on the earth's surface.

These variations in the weight of a body and in the value of g due to latitude, elevation, and other causes are less than one half of one per cent and are rarely considered in engineering problems. But when precision is desired, and also when the fundamental units of weight, force, and mass are under consideration, these variations in the weight of a body and in the value of g are essential.

A standard locality has been fixed by international conference. At the standard locality the value of g is 32.1740 feet per second

per second. For convenience in computation the value $g = 32.2$ has been used in the text.

10. Newton's laws of motion. Many of the principles of mechanics are contained in the three following laws, first stated by Sir Isaac Newton:

I. Every body continues in a state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

II. The rate of change of momentum is proportional to the force applied and takes place in the direction of the straight line in which the force acts.

III. To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed.

The *first law* contains a statement of the inertness of matter and an implied definition of force.

Inertia is a certain characteristic property of matter by which it remains in its state of rest or of motion in a straight line unless acted upon by force. Matter does not of itself exert force; it reacts against a force applied to change its state of rest or of motion. The primitive idea of inertia arises through the muscular effort necessary to set a body in motion or to retard its motion.

The definition of force implied in this law is as follows: *Force is any cause which alters or tends to alter a body's state of rest or of uniform motion in a straight line.*

The *second law* asserts that the force acting is proportional to the time rate of change of momentum.

The *momentum* of a body is defined as the product of the mass of the body and its velocity.

The *mass* of a body is a measure of its inertness or tendency to resist the action of force. Hence the force

$$F = k\frac{d}{dt}(mv) = kma,$$

where m is the constant mass, v is the velocity, and a is the acceleration.

The following idealized experiment may aid in forming a concept of mass: The acceleration imparted to a cubic foot

of lead resting on a smooth horizontal plane by a constant horizontal force is found to be different from the acceleration imparted to a cubic foot of cork by an equal force under the same circumstances. Since the forces are the same in both cases, and since no other forces are acting — friction being eliminated and the force of gravity being perpendicular to the direction of the constant forces — it follows that the different accelerations must be due to some inherent characteristic of the bodies. This characteristic of the bodies is called *mass*. If the accelerations of the lead and the cork are p and q respectively, the mass of the lead is $\frac{q}{p}$ times the mass of the cork. In other words, *the ratio of the masses of two bodies is inversely proportional to the accelerations produced in them by any constant force applied to them in succession.*

Experiments verify the following implication of the second law: Each force acting on a body produces a corresponding acceleration independent of other forces acting or of the motion of the body, and the acceleration is in the direction of the force.

The *third law* asserts that forces always occur in pairs. A *single* force does not exist. A force is one aspect of the mutual action between two bodies. A book resting on a table exerts a force *upon the table*; the table exerts an equal upward force *upon the book*. The earth exerts an attractive force *upon the book*, and the book also exerts an equal attractive force *upon the earth*. When a boy throws a ball, his fingers exert a force upon the ball; the ball exerts an equal and opposite force upon his fingers. The equality of action and reaction is sometimes given as a reason why the ball should not move at all. Clearly the ball moves because only one of the forces constituting the mutual action between the ball and the fingers acts upon the ball. If both the action and reaction acted upon the ball, it would not move.

11. Systems of units. Two systems of units * are in common use. They are the British Absolute System and the Engineer's Gravitational System. In the first system *mass*, *length*, and *time* are taken as the fundamental quantities, and in the second system *force*, *length*, and *time* are taken as the fundamental

* *Bulletin of the American Physical Society*, Report No. 3, and also Barton's "Analytical Mechanics."

quantities. For simplicity the proportionality factor, k, in the fundamental equation of mechanics, $F = kma$, is taken as unity in both systems, thereby making the fundamental equation

$$F = ma.$$

Both systems use the same unit acceleration, namely, one foot per second per second. In order that this equation be satisfied *numerically*, it is evident that both the unit mass and the unit force cannot be chosen arbitrarily but that they must be chosen so that the unit force produces unit acceleration in the unit mass. If the unit mass is chosen arbitrarily, the unit force is derived from or defined by the fundamental equation. If the unit force is chosen arbitrarily, the unit mass is defined by the fundamental equation.

12. British Absolute System. In the British Absolute System of units the unit mass, called a *pound*, is taken as the mass of a certain cylinder of platinum arbitrarily chosen, and established as the standard by act of the British Parliament. The unit force is called a *poundal* and is defined, according to the equation $F = ma$, as that force which acting alone upon the unit mass, one pound, imparts to it an acceleration of one foot per second per second.

It is convenient to compare the unit force, a poundal, with the weight of a unit mass at the standard locality. Since the force of gravitation acting upon the unit mass, one pound, at the standard locality imparts to that mass an acceleration of 32.1740 feet per second per second, the poundal is $\frac{1}{32.1740}$ of the weight of the unit mass at the standard locality.

13. Engineer's Gravitational System. In the Engineer's Gravitational System of units the unit force, called a *pound*, is taken as the earth pull or weight of the unit mass of the British absolute system* at the standard locality, where g is equal to 32.1740 feet per second per second. According to the equation $F = ma$, the unit mass, sometimes called a *slug*, is defined as that mass which when acted upon by the unit force, one pound, has imparted to it an acceleration of one foot per second per second.

*Unit mass, one pound, of the British absolute system or its equivalent, $\frac{1}{2.204622}$ kilogram mass of the metric system.

Since the unit force, one pound, gives a mass of one pound an acceleration of 32.1740 feet per second per second, it follows that the derived unit mass, one slug, is 32.1740 times the unit mass of the British Absolute System. Hence the mass of a body may be obtained by dividing its weight at the standard locality by 32.1740. Moreover the mass of a body may be found by dividing its local weight by the local value of g. For when a body is subject to gravity alone, the force acting is the local weight W_1, and the acceleration is the local acceleration g_1. The fundamental equation $F = ma$ becomes, in this case,

$$W_1 = mg_1$$

or

$$m = \frac{W_1}{g_1}.$$

Since the mass m does not change with its location, it follows that the ratio $\frac{W_1}{g_1}$ does not change.

When this ratio is substituted for m in the fundamental equation $F = ma$, the equation takes the form

$$F = \frac{W_1}{g_1} a.$$

This equation, in which g is taken as 32.2, is most frequently used.

The Engineer's Gravitational System of units is used in the text, and the value of g is taken as 32.2 feet per second per second.

14. Measurement of mass, force, and weight. Usually it is not convenient to measure directly the acceleration of a moving body, and therefore it is not easy to compare different masses by comparing the accelerations imparted to them by equal forces. It is equally difficult to measure a force by measuring the acceleration which it imparts to a given mass. Actually forces and masses are measured by comparing them with the earth's pull on certain bodies known as standard weights. This is done either by *lever scales* or *spring scales*.

The unit force, one pound, is defined as the weight of a certain cylinder of platinum at the standard locality where $g = 32.1740$. Weights that are calibrated in the standard locality by means of this platinum cylinder are referred to as *standard weights*.

They are not standard weights, however, in the sense that their weight remains constant, because the weight of any body changes with the locality. The mass of the standard weight is invariable, however, and therefore standard weights are in reality *standard masses.*

Measurement of masses. A merchant using *lever scales* and standard weights gives the same amount of salt per pound to his customer, independent of the locality. If he uses a *spring* scale, which has been calibrated with standard weights at the standard locality, he gives too much or too little, depending on the new locality.

Measurement of forces. On the contrary, a definite force — a certain fixed pull — will give the same reading on a *spring* scale, independent of the locality. This same pull measured on *lever* scales with standard weights will give different results, depending on the locality.

Although the weight of a body is not constant, yet it remains the most convenient and practical approximate measure of mass and also of force.

Moreover all determinations of mass, force, or weight, in whatever locality they are made or whatever kind of scales are used, may easily be reduced to their correct value in the standard locality by the use of the local value of g.

PROBLEMS

1. Determine the sum of the vectors (12 lb., 30°) and (18 lb., 0°).
Ans. 29 lb., 12°.

2. Find the vector which results from subtracting the vector (150 lb., 15°) from the vector (100 lb., 60°). *Ans.* 106 lb., 153° 16′.

3. Show that the sum of the vectors represented by the medians of any triangle all drawn from the vertices toward the opposite sides is zero.

4. A particle moves around a circle of radius 6 ft. with a uniform speed of 8 ft. per second. What is the displacement from the starting point at the end of 5 sec.? What is the velocity at the end of 5 sec.? *Ans.* 2.29 ft., 101°; 8 ft. per second, 112°.

5. A particle moves along a straight line in such a manner that its distance s from a fixed point on the line is represented by the equation $s = 4\,t^3 + t^2$, where t represents the time in seconds. Find the velocity and the acceleration of the particle at the end of 4 sec.
Ans. $v = 200$ ft. per second, $a = 98$ ft. per second per second.

6. A force of 20 poundals acts upon a mass of 3 lb. What acceleration is produced? *Ans.* $a = 6.67$ ft. per second per second.

7. A force of 20 lb. acts upon a mass of 3 slugs. What acceleration is produced? *Ans.* $a = 6.67$ ft. per second per second.

8. How much will a standard 50-pound weight weigh where $g = 32.2$? *Ans.* 50.04 lb.

9. A grocer buys 1000 lb. of sugar at the standard locality and also a spring scale calibrated at the standard locality. What does the sugar weigh on his spring scale, where $g = 32.12$? *Ans.* 998.3 lb.

10. A body has an acceleration of 900 mi. per hour per minute. Express its acceleration in feet per second per second.

Ans. 22 ft. per second per second.

STATICS

CHAPTER II

CONCURRENT FORCES

15. Introduction. The concept of force is common to everyone through the experience of pushing, pulling, and lifting. The fundamental characteristics of force are magnitude, direction, and point of application. Force is therefore a vector. The vectorial representation of a force by a fixed, directed, straight line is very convenient and is frequently used.

The line of action of a force is the line through its point of application in the direction of the force. It is taken as axiomatic that a force may be applied at any point in its line of action without in any way altering its effect. This is known as the *principle of the transmissibility of force.*

A *distributed* force is one which acts over a surface or throughout a volume. Strictly speaking, all forces are distributed forces. It is convenient, however, to regard many forces which are distributed over a relatively small area as concentrated at some point of the area.

A problem in statics is concerned with a *body* or *several bodies* that are acted upon by forces. Frequently the problem is idealized, the body is lost sight of, and the problem becomes one of the analysis, composition, and reduction of *forces,* without any reference to the *body* upon which the forces act.

A system of forces is any number of forces which are considered collectively, that is, a group of forces. Any system of forces which may be substituted for another system of forces without any change of effect is called an equivalent system of forces. The *simplest* equivalent system is called the *resultant* of the system. In particular, when a system of forces acts *at a point,* the simplest equivalent system, or the resultant, is a single force acting at the point. When the single force or

resultant is zero, the system of forces is said to be in *equilibrium.* If a system of forces acting at a point is in equilibrium, there is no tendency to change the condition of rest or of motion of the point.

A fundamental problem of statics is to determine the necessary and sufficient conditions that a system of forces may be in equilibrium. In the solution of problems in statics the processes of mathematical analysis, as well as graphical or vector operations, will be employed.

For simplicity the various systems of forces are classified and treated under the three following heads:

(1) Systems of concurrent forces.
(2) Systems of parallel forces.
(3) Systems of nonconcurrent forces.

A system of *concurrent* forces is a system of forces whose lines of action all pass through a common point.

A system of *parallel forces* is a system of forces whose lines of action are all parallel.

A system of *nonconcurrent forces* is a system of forces whose lines of action do not pass through a common point.

The lines of action of the forces in each of the groups may all lie in a plane or in space.

The remainder of this chapter is devoted to the study of systems of concurrent forces.

16. Resolution and composition of concurrent forces; parallelogram of forces. Since forces are vectors they can be combined and resolved by the processes of vector analysis. Let OA and OB be the vector representatives of any two forces P and Q respectively. The vector sum of the two vectors OA and OB is the vector OC, the diagonal of the parallelogram constructed on OA and OB as sides. The vector OC represents a force, R, which is the vector sum of the two forces P and Q. The force R is an exact equivalent in every sense of the two forces P and Q, and hence the two forces P and Q may be replaced by the single force R. The force R is called the *resultant* of the two forces P and Q.

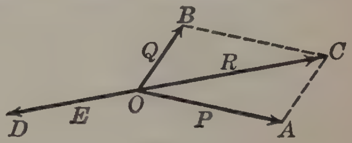

FIG. 7

The *law of the parallelogram of forces* may be taken as axiomatic or verified by experiment. It may be stated as follows: If OA and OB represent any two forces, P and Q, in magnitude and direction, the diagonal OC of the parallelogram constructed on OA and OB as sides represents the resultant force R of the forces P and Q in magnitude and direction.

A force E equal in magnitude to R acting at O in the direction opposite to OC is called the equilibrant of P and Q. The equilibrant E and the resultant R are evidently in equilibrium. The forces P, Q, and E form a system of forces in equilibrium, and any one of them is the equilibrant of the other two.

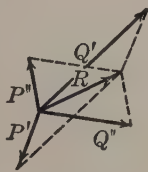

FIG. 8

The process of combining two forces, P and Q, to give a resultant R which may replace them is called the *composition of forces*. The reverse process of resolving any force R into components P and Q is called the *resolution* of a force into its component forces. It is evident from Fig. 8 that any force R can be resolved into two components in many ways.

PROBLEMS

1. Find the resultant of two forces, of 20 lb. and 30 lb., which make an angle of 45° with each other.
Ans. $R = 46.4$ lb. at an angle of 17° with the 30-pound force.

2. Find the resultant of two forces, of 20 lb. and 30 lb., which make an angle of 150° with each other.
Ans. $R = 16.1$ lb. at an angle of 38° 15′ with the 30-pound force.

3. Find the resultant of two forces, each 75 lb., which make an angle of 120° with each other.
Ans. $R = 75$ lb. at an angle of 60° with either force.

4. Resolve a force of 40 lb. into two components, one of which shall be 400 lb. and make an angle of 135° with the 40-pound force.
Ans. 429 lb. at an angle of 41° 13′ with the 40-pound force.

5. If three concurrent forces are in equilibrium, show that their lines of action must lie in a plane.

6. If three coplanar forces are in equilibrium, show that their lines of action are concurrent or parallel.

17. Triangle of forces. In many problems simplicity recommends the use of the triangle of forces instead of the parallelogram of forces. Any two forces P and Q, together with their

resultant R and their equilibrant E, may be represented by the triangle OAC, due attention being given to the pointing of the arrows and to the fact that all the forces are *concurrent.*

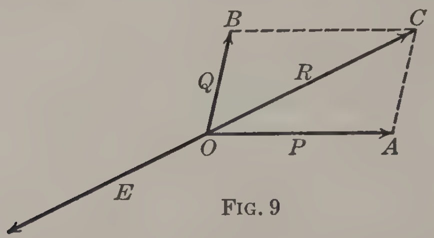

FIG. 9

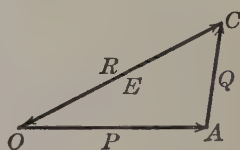

FIG. 10

Referring to Fig. 10, the force P is represented by OA and the force R by OC. The force E may be represented by CO. Now AC may represent the force Q in magnitude and direction, but it must be borne in mind that the force Q acts at O and not at A.

Another view of the triangle of forces may be presented by the introduction of the *space diagram* and the *force diagram.* Let P and Q be any two forces and E their equilibrant as before, so that the three forces P, Q, and E form a system of forces in equilibrium.

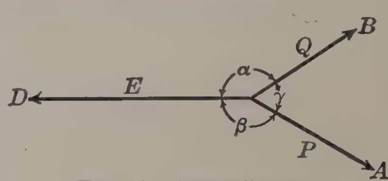

FIG. 11. Space diagram

The *space diagram* shows the point of application of the forces and their directions but not necessarily their magnitudes. The *force diagram* shows their magnitudes (to scale) and their directions but not the point of application.

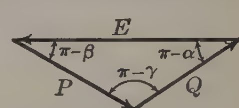

FIG. 12. Force diagram

It is evident that if any three concurrent forces are in equilibrium, their vector representatives form a triangle. *The arrows follow each other around the triangle.* Conversely, if the vector representatives of three concurrent forces form a triangle and the arrows follow each other around the triangle, the three concurrent forces are in equilibrium.

From the Sine Law and the force diagram it is evident that

$$\frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{E}{\sin \gamma}.$$

It follows, therefore, that if three concurrent forces, all acting outward or all acting inward, are in equilibrium, the ratio of

each force to the sine of the angle between the other two forces is constant. Conversely, if three concurrent forces, all acting outward or all acting inward, are proportional to the sines of the angles between the other two, the three forces are in equilibrium.

Resolution of a force into components. It is evident that any force R can be resolved into components P and Q, the sole requirement being that P and Q form the two sides of a triangle of which the given force is the third side.

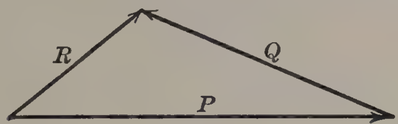

FIG. 13

EXAMPLES

1. A body weighing 60 lb. is suspended by a cord from a small ring which is supported by two cords making angles of 60° and 20° with the horizontal. Determine the tension in the cords.

Solution. Graphical method. The first step is to draw the space diagram showing the ring at C, the cord CD supporting the weight, the cords CA and CB connecting the ring with the fixed points A and B. The tensions in the strings CA, CB, and CD form a system of forces in equilibrium, concurrent at C.

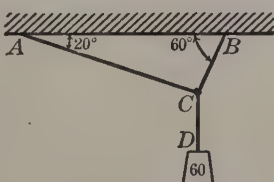

FIG. 14. Space diagram

The force diagram is constructed by first drawing a vector lm to represent the known tension in CD; that is, lm is drawn parallel to CD, and its length represents to any convenient scale a force of 60 lb. Then through l draw a line ln parallel to the cord AC, and finally through m draw a line mn parallel to cord BC to intersect ln at n. Since the tension in CD acts downward from C, the vector lm must be directed downward. Also since the forces are in equilibrium, the arrows must follow each other around the force diagram. The vectors mn and nl in the force diagram represent in magnitude and direction to the same scale the tensions in CB and CA respectively.

Analytic method. The directions of the tensions in the cords AC and BC are known from the space diagram. The magnitudes of the tensions are found from the equations

$$\frac{P}{\sin 70^\circ} = \frac{Q}{\sin 30^\circ} = \frac{60}{\sin 80^\circ}.$$

Solving these equations, $P = 57.3$ lb. and $Q = 30.5$ lb.

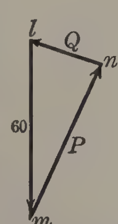

FIG. 15. Force diagram

2. A body weighing 20 lb. rests against the smooth convex surface of a fixed circular cylinder whose axis is horizontal, being held by a string which rests upon the cylinder and sustains a 10-pound weight. Determine the position of equilibrium.

Solution. Make a sketch of the problem, as shown, where θ represents the unknown angle which determines the position of the 20-pound weight. This weight is to be in equilibrium under the action of the following forces: (1) the action of gravity; (2) the force exerted by the cylinder, which, since the cylinder is smooth, acts along the radius through the 20-pound weight; (3) the tension in the string, which acts perpendicularly to this radius.

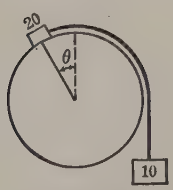

FIG. 16. Sketch

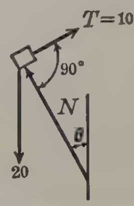

FIG. 17. Space diagram

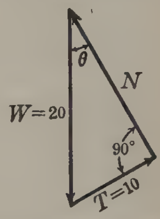

FIG. 18. Force diagram

Make a space diagram showing the body and the forces acting upon it. Draw the force diagram. Since the force diagram is a right triangle, $\sin\theta = \frac{10}{20} = \frac{1}{2}$, and hence $\theta = 30°$.

PROBLEMS

1. Find the horizontal force required to support a weight of 8 lb. upon a smooth inclined plane of length 10 ft. and height 6 ft. *Ans.* 6 lb.

2. Three smooth pegs are driven into a wall at the vertices of an equilateral triangle having its base horizontal. A string passing over the three pegs carries a weight of 100 lb. at each end. Calculate the tension in the string and the pressure on each peg.
Ans. 100 lb., 173.2 lb., 51.8 lb.

3. A weight of 100 lb. rests on a smooth inclined plane of length 25 ft. and height 7 ft. Find the force parallel to the plane necessary to prevent the body from sliding down the plane. *Ans.* 28 lb.

4. One end of a pin-connected truss is shown in Fig. 19. The upward reaction of the foundation against it is 100,000 lb. The space diagram is shown in Fig. 20. Find the forces in AB and AC, and determine whether the members are in tension or compression.
Ans. 115,500 lb., 57,750 lb.

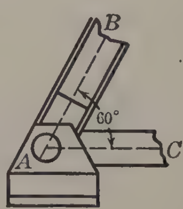

FIG. 19

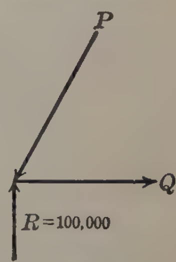

FIG. 20

5. A wheel 26 in. in diameter weighs 10,000 lb. What horizontal force applied at the axle is necessary to start the wheel over a block 1 in. high? *Ans.* 4167 lb.

6. One end of a beam 12 ft. long is hinged to a wall; the other end is supported by a cable which is attached to the wall at a point 2 ft. above the hinge, as shown in Fig. 21. The outer end of the beam supports a load of 1000 lb. Find the stress in the beam and also that in the cable, neglecting the weight of the beam and cable.

Ans. 6000 lb., 6083 lb.

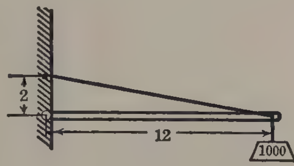

FIG. 21

7. A smooth sphere is supported in contact with a smooth vertical wall by a string fastened to a point on its surface, the other end being attached to a point on the wall. If the length of the string is equal to the radius of the sphere and the weight of the sphere is 16 lb., find the inclination of the string to the vertical, the tension of the string, and the reaction of the wall. *Ans.* 30°, 18.48 lb., 9.24 lb.

8. A particle may be held in equilibrium on a smooth inclined plane either by a horizontal force of 60 lb. or by a force of 50 lb. making an angle of 30° with the plane. Find the inclination of the plane to the horizontal and the weight of the particle.

Ans. 43° 15′, 62.5 lb.

18. Polygon of forces. The principle of the triangle of forces admits of easy extension to find the resultant or equilibrant of any number of concurrent forces, or to resolve any force into any number of concurrent components.

Let P_1, P_2, P_3, and P_4 be any forces acting at a point O, their directions being shown in the space diagram. Starting from any point O', construct a force diagram by laying off $O'A$ equal to P_1 (to scale) and parallel to it. From A lay off AB equal and parallel to P_2; then $O'B$ is the resultant of P_1 and P_2. From B lay off BC equal and parallel to P_3; then $O'C$ is the resultant of R_1 and P_3, or the resultant of P_1, P_2, and P_3. From C lay off CD equal and parallel to P_4; then $O'D$ is the resultant of R_2 and P_4 and hence the resultant of P_1, P_2, P_3, and P_4. The magnitude of the resultant is obtained by scaling $O'D$ in the force diagram. The resultant acts, of course, at the common point of the concurrent forces, O, in the space diagram. The direction of the resultant R is OR, parallel to $O'D$. Attention must be given to the direction

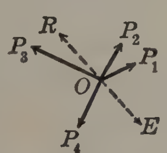

FIG. 22. Space diagram

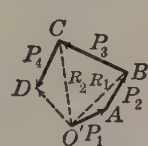

FIG. 23. Force diagram

of the forces as indicated by the arrows. The resultant always points from the beginning O' to the last point of the last force. The equilibrant of P_1, P_2, P_3, and P_4 is represented in line of action and point of application by OE and in magnitude by DO'. It is self-evident that the forces P_1, P_2, P_3, P_4 and E form a system in equilibrium.

19. Conditions for equilibrium; graphical method. The condition that any number of concurrent forces are in equilibrium is that the force polygon must close. Conversely, if the force polygon closes, the concurrent forces must be in equilibrium.

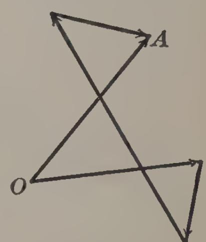

FIG. 24

Any force P (OA) may be resolved into any number of concurrent forces arbitrarily by drawing a force diagram at pleasure, the sole requirement being that the polygon begin at O and end at A and that the arrows travel from O around the chosen polygon to A.

The method of § 18 and its deductions hold for concurrent forces in *space*.

20. Analytic method; rectangular components of a force. The composition or reduction of concurrent *coplanar* forces by means of the force polygon is simple and convenient when carried out by graphical processes. Any graphical process, however, is limited in its accuracy. Of course, any desired accuracy may be obtained by solving trigonometrically each triangle in the graphical process, but the analytic method requires less computation.

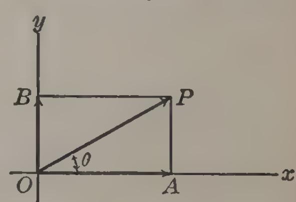

FIG. 25

In the analytic method each force is first replaced by its rectangular components, as follows: Let OP represent in magnitude, direction, and point of application any one of a system of forces P_1, P_2, $\cdots$ concurrent at O. Through O draw any two rectangular axes, and let θ be the angle which the force P makes with the x axis. Evidently the force P may be resolved into the two components OA and OB, where $OA = P\cos\theta$ and $OB = P\sin\theta$. The force P is then replaced by its two rectangular components, $P\cos\theta$ and $P\sin\theta$. Frequently the x axis is assumed horizontal and the y axis vertical; the rectangular

components $P \cos \theta$ and $P \sin \theta$ are then called the horizontal and vertical components, respectively, of the force P. For brevity, $P \cos \theta$ and $P \sin \theta$ are usually written X and Y respectively and are called the x and y components of the force P.

In the same way each one of the forces $P_1, P_2, \cdots$ is replaced by its rectangular components. Hence the system of concurrent forces $P_1, P_2, \cdots$ acting at O and making angles $\theta_1, \theta_2, \cdots$, respectively, with the x axis is equal to and is replaced by the following system:

$$P_1 \cos \theta_1,\ P_2 \cos \theta_2, \cdots \text{ along the } x \text{ axis},$$
$$P_1 \sin \theta_1,\ P_2 \sin \theta_2, \cdots \text{ along the } y \text{ axis}.$$

Since $P_1 \cos \theta_1, P_2 \cos \theta_2, \cdots$ lie in the same line, they may be added algebraically, the result being a single force $P_1 \cos \theta_1 + P_2 \cos \theta_2 + \cdots$ along the x axis. This last expression may be written $X_1 + X_2 + \cdots$ and is usually abbreviated into ΣX, where the Greek letter Σ (sigma) is used to mean the sum of the x components. The sum of the y components is written ΣY.

The original forces $P_1, P_2, \cdots$ are finally replaced by ΣX along the x axis and ΣY along the y axis. The forces ΣX and ΣY are now combined by the Parallelogram Law into a single resultant R, where the magnitude of R is given by the equation

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2,$$

and the direction of R is given by the equation

$$\tan \theta = \frac{\Sigma Y}{\Sigma X}.$$

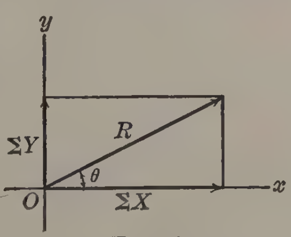

FIG. 26

It is to be observed that the x or y components of some of the forces may be negative. This may be determined from the sign of the functions of the angles $\theta_1, \theta_2, \cdots$, but it is simpler to use the smallest angle that the given force makes with the x axis, and by observation of a reasonably accurate space diagram to determine the algebraic signs of the components.

This method is easily extended to any number of concurrent forces *in space*. Each force P_1 is replaced by $P_1 \cos \alpha_1$ along the x axis, $P_1 \cos \beta_1$ along the y axis, and $P_1 \cos \gamma_1$ along the

z axis, where α_1, β_1, γ_1 are the direction angles of the force P_1. The resultant force R is given by the equation

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$

where
$$\Sigma X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \cdots,$$
$$\Sigma Y = P_1 \cos \beta_1 + P_2 \cos \beta_2 + \cdots,$$
and
$$\Sigma Z = P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \cdots.$$

The direction of the resultant R is given by its direction cosines, thus:

$$\cos \alpha = \frac{\Sigma X}{R}, \quad \cos \beta = \frac{\Sigma Y}{R}, \quad \cos \gamma = \frac{\Sigma Z}{R}.$$

21. Conditions of equilibrium; analytic method. If a system of concurrent forces is in equilibrium, the resultant of the system must vanish. Analytically this requires that

$$(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 = 0.$$

But since $(\Sigma X)^2$, $(\Sigma Y)^2$, and $(\Sigma Z)^2$ are each positive, this necessitates that $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma Z = 0$.

In the case of coplanar forces the conditions for equilibrium are evidently

$$\Sigma X = 0 \quad \text{and} \quad \Sigma Y = 0.$$

EXAMPLES

1. Find the resultant of the following concurrent forces: a force of 10 lb. passing through the origin and making an angle of 30° with the positive x axis; a force of 5 lb. at 90°; a force of 15 lb. at 135°; a force of 20 lb. at 240°; and a force of 12 lb. at 300°.

Solution. Analytic method. The work is facilitated by arranging it in tabular form.

Force	Angle	Horizontal Component	Vertical Component
10	30°	$10 \cos 30° = 8.66$	$10 \sin 30° = 5.00$
5	90°	$5 \cos 90° = 0.00$	$5 \sin 90° = 5.00$
15	135°	$-15 \cos 45° = -10.61$	$15 \sin 45° = 10.61$
20	240°	$-20 \cos 60° = -10.00$	$-20 \sin 60° = -17.32$
12	300°	$12 \cos 60° = 6.00$	$-12 \sin 60° = -10.39$
		$\Sigma X = -5.95$	$\Sigma Y = -7.10$

$$R = \sqrt{(-5.95)^2 + (-7.10)^2} = 9.26 \text{ lb.}$$

$$\tan \theta = \frac{\Sigma Y}{\Sigma X} = \frac{-7.10}{-5.95} = 1.193.$$

$$\therefore \theta = 230° \, 3'.$$

Graphical method. The vectors representing the given forces are combined by the method of the polygon of forces. The forces may be taken in any order. The resultant is represented by the vector OE, the closing line of the polygon. Its magnitude and direction may be measured on the force diagram.

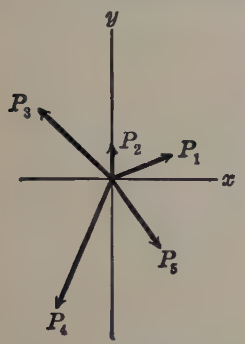

FIG. 27. Space diagram

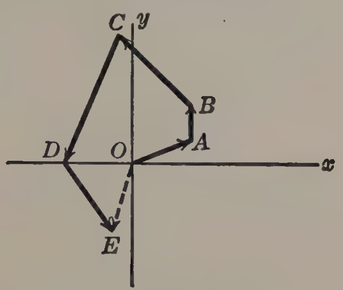

FIG. 28. Force diagram

2. A weight of 100 lb. suspended by a string is pulled aside by a horizontal force P so that the string makes an angle of 30° with the vertical. Find the tension in the string and the horizontal force P.

Solution. Analytic method. Take the origin at the point O, where the lines of action of the forces meet. The conditions of equilibrium, $\Sigma X = 0$ and $\Sigma Y = 0$, give $P - T\cos 60° = 0$ and $T\cos 30° - 100 = 0$.

Solving these simultaneous equations,

$$T = \frac{100}{\cos 30°} \quad \text{and} \quad P = \frac{100 \times \cos 60°}{\cos 30°},$$

or

$$T = 115.5 \text{ lb.} \quad \text{and} \quad P = 57.7 \text{ lb.}$$

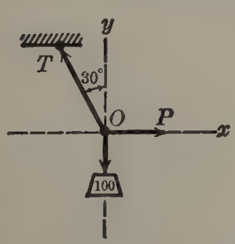

FIG. 29

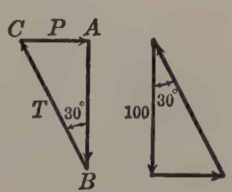

FIG. 30

Graphical method. Draw the vector AB to represent the 100-pound weight. Through either extremity of AB draw a line parallel to the force P and through the other extremity draw a line parallel to the force T. Produce these lines to their intersection C. The vectors CA and BC represent the forces P and T respectively.

By the triangle of forces. From the figure,

$$\frac{P}{\sin 30°} = \frac{T}{\sin 90°} = \frac{100}{\sin 60°},$$

from which $P = 57.7$ lb. and $T = 115.5$ lb.

3. A weight of W lb. is mounted on a tripod whose legs are each l ft. in length. The feet of the tripod rest on the ground at the vertices of an equilateral triangle whose sides are each a ft. in length. Find the thrust in each leg.

Solution. From symmetry it is evident that the thrust in each leg is the same. The component of each thrust along the vertical OZ is $P \cos \alpha$, where α is the angle between each leg and the vertical. For equilibrium of the forces acting at O it is necessary that

$$3 P \cos \alpha = W. \tag{1}$$

From the figure,

$$\cos \alpha = \frac{OZ}{OK} = \frac{\sqrt{\overline{OK}^2 - \overline{ZK}^2}}{OK} = \frac{\sqrt{l^2 - \frac{a^2}{3}}}{l}, \tag{2}$$

since ZK is two thirds of the median of the equilateral triangle. Hence, from (1),

$$P = \frac{Wl}{3\sqrt{l^2 - \frac{a^2}{3}}} = \frac{Wl}{\sqrt{9\, l^2 - 3\, a^2}}. \tag{3}$$

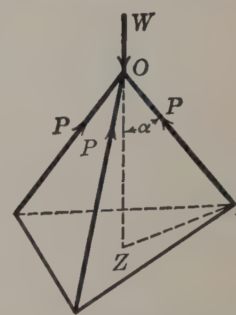

FIG. 31

4. A particle of weight W is in equilibrium under the action of gravity, a horizontal force kW, a force F making an angle θ with the horizontal, and a force N at right angles to F. Find F and N in terms of W, k, and θ.

Solution. It is frequently advantageous to place the x and y axes in a position other than the horizontal and vertical directions. In this problem the axes are taken along the unknown forces F and N, since the conditions of equilibrium yield equations which separately contain only one of the unknown forces, and thereby the necessity of solving two simultaneous equations is obviated.

The condition $\Sigma X = 0$ gives

$$F + kW \cos \theta - W \sin \theta = 0.$$

Hence $$F = W(\sin \theta - k \cos \theta).$$

The condition $\Sigma Y = 0$ gives

$$N - kW \sin \theta - W \cos \theta = 0.$$

Hence $$N = W(k \sin \theta + \cos \theta).$$

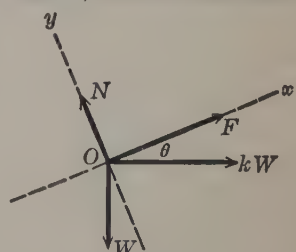

FIG. 32

PROBLEMS

1. A rope-walker weighing 160 lb. stands at the middle of a wire cable 75 ft. long. The center of the cable is depressed 4 ft. below the level of the end supports. Determine the tension in the cable, neglecting its weight. *Ans.* 750 lb.

2. Solve Example 4 above by resolving the forces horizontally and vertically.

3. The pin-jointed framework shown in Fig. 33 is supported by two vertical posts and carries a load of 300 lb. at its vertex. Find the stress in AB, AC, and AD. *Ans.* 335 lb., 300 lb., and 150 lb.

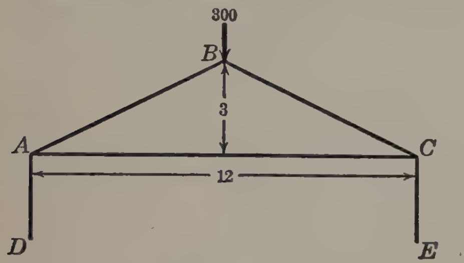

FIG. 33

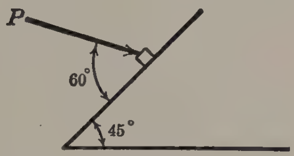

FIG. 34

4. A block weighing 120 lb. rests upon a smooth inclined plane as shown in Fig. 34. What force P is required to hold the block stationary? What is the pressure between the block and the plane?

Ans. 169.7 lb., 231.8 lb.

5. Steam in the cylinder of a locomotive exerts a pressure of 75,000 lb. on the piston. What is the thrust in the connecting rod when the crank makes an angle of 60° with the horizontal, and what is the normal pressure upon the guides, assuming that the engine is not running? *Ans.* 75,800 lb., 10,940 lb.

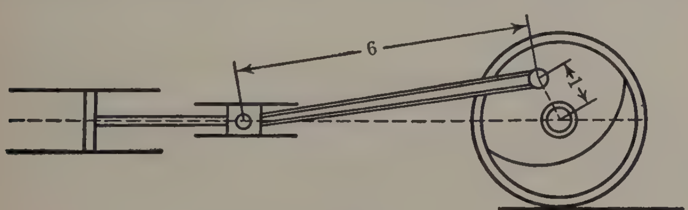

FIG. 35

6. A weight of 10 lb. is supported on a smooth inclined plane making an angle of 30° with the horizontal by a string which makes an angle of 15° with the plane. Find the tension in the string and the reaction of the plane on the weight. *Ans.* 5.18 lb., 7.32 lb.

7. Two weights, A and B, weighing 100 lb. each, are suspended from a fixed point C by two strings, AC and BC, each 10 ft. long. The weights are separated by a light stick of length 4 ft. Find the tension in the strings and the thrust in the stick. *Ans.* 102.1 lb., 20.4 lb.

8. Three light rods, each 6 ft. long, are jointed together to form an equilateral triangle ABC. The rod AB is fixed in a horizontal position, and a weight of 100 lb. is suspended from the lower vertex C. Find the tensions in the rods AC and BC. If the rod AB is inclined at an angle of 30° with the horizontal, find the tensions in AC and BC.

Ans. 57.7 lb., 100 lb.; 0 lb.

9. A smooth circular wire of radius 2 ft., situated in a vertical plane, carries a small ring of weight 10 lb., as shown in Fig. 36. The ring is held in position on the wire by a string 1 ft. long attached at the extremity of a horizontal diameter. Find the tension in the string and the pressure between the wire and ring.

Ans. 9.04 lb., 2.58 lb.

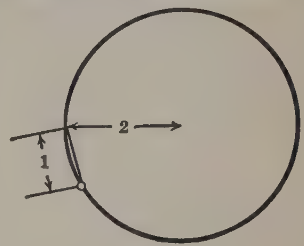

FIG. 36

10. The smooth pin shown at A in Fig. 37 fastens four members of a bridge truss together and supports a load of 16,000 lb. Find the stresses F_1 and F_2, and determine whether they act toward or away from the pin at A.

Ans. $F_1 = 66{,}630$ lb. away from pin, or tension;
$F_2 = 15{,}960$ lb. toward pin, or compression.

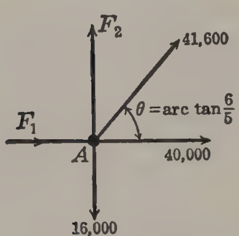

FIG. 37

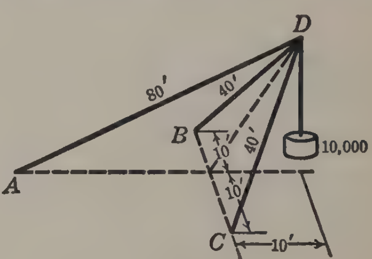

FIG. 38

11. Find the forces acting upon each member of the crane shown in Fig. 38. The weight of the members is neglected.

Ans. $AD = 3520$ lb., $BD = CD = 6230$ lb.

12. Three equal smooth spheres are lying in contact on a horizontal plane and are held together with a string. An equal sphere of weight 9 lb. rests upon the three spheres. Find the tension in the string, assuming that the weight of the upper sphere is concentrated at its center.

Ans. 1.22 lb.

13. Six equal cylinders, weighing 1000 lb. each, are piled as shown in Fig. 39. Find the pressures at A, B, C, D, E, and F.

Ans. 577 lb., 2000 lb., 2000 lb., 1155 lb., 577 lb., 577 lb.

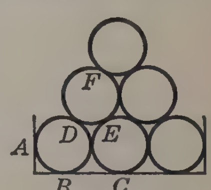

FIG. 39

14. A string attached to a fixed point A and passing over a smooth peg B carries at the free end a weight P, as shown in Fig. 40. A weight W is suspended at a point C by means of a knot. The dis-

tance AB is given as a, and the length of the string AC is b. It is required to find the value of θ for equilibrium.

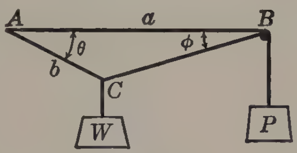

FIG. 40

HINT. From the space diagram,

$$a : \sin(\theta + \phi) = b : \sin \phi ;$$

and from the force diagram,

$$P : W = \cos \theta : \sin(\theta + \phi).$$

$$Ans.\ \cos^3 \theta - \frac{P^2a^2 + W^2(a^2 + b^2)}{2\,abW^2} \cos^2 \theta + \frac{P^2a}{2\,bW^2} = 0.$$

15. A particle of weight 4 lb. resting against a smooth circular cylinder of radius 2 ft. whose axis is horizontal is held in equilibrium by a string passing over a smooth pulley 4 ft. above the axis of the cylinder, as shown in Fig. 41. The string supports a weight of 3 lb. Find the angle θ which the radius through the 4-pound weight makes with the vertical and the pressure between the 4-pound weight and the cylinder.

Ans. arc cos $\frac{11}{16} = 46° 34'$, $N = 2$ lb.

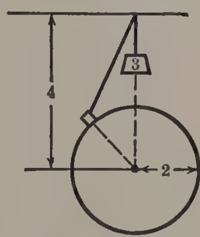

FIG. 41

16. A rope 77 ft. long has its ends attached at two points, A and B, 63 ft. apart on the ceiling of a room. The rope passes through a smooth ring weighing 100 lb. Find the horizontal force which must be applied to the ring to keep it in equilibrium in a position 20 ft. below the ceiling and in a vertical plane through A and B. Also find the tension in the rope. If the ring is fastened to the rope in this position and the horizontal force removed, find the tensions in the rope. *Ans.* 27.3 lb., 84.4 lb., 61.9 lb., 95.2 lb.

17. The pin-jointed framework in Fig. 42, attached to the wall by pins at A and D, is loaded as shown. Find the stress in each member. Find also the horizontal and vertical components of the forces exerted by the wall on the pins at A and D.

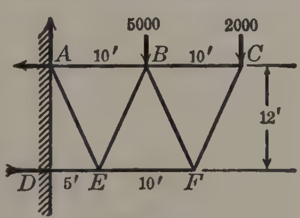

FIG. 42

Ans. $BC = 833$ lb. tension,
$CF = 2167$ lb. compression,
$BF = 2167$ lb. tension,
$EF = 1667$ lb. compression,
$BE = 7583$ lb. compression,
$AB = 4583$ lb. tension,
$AE = 7583$ lb. tension,
$DE = 7500$ lb. compression.

Horizontal component exerted by wall at $D = 7500$ lb. push.
Vertical component exerted by wall at $D = 0$.
Horizontal component exerted by wall at $A = 7500$ lb. pull.
Vertical component exerted by wall at $A = 7000$ lb. upward.

CHAPTER III

MOMENT OF A FORCE—PARALLEL FORCES

22. Moment of a force. The moment of a force about any point is the product of the magnitude of the force and the perpendicular distance from the point to the line of action of the force. The simple concept of the moment of a force about a point is extremely useful and is of wide application. The moment of the force F about the point O is $F \times p$ and is usually measured in pound-feet. The moment of a force is a measure of the turning effect or tendency of the force to rotate the body to which the force is applied. A moment is considered positive or negative according as it tends to produce positive or negative rotation in the trigonometric sense. Thus, in Fig. 43 the moment is positive. In general, the moment of a vector about an *axis* may be defined as the moment of the projection of the vector upon a plane perpendicular to the axis about the point of intersection of the axis and the perpendicular plane.

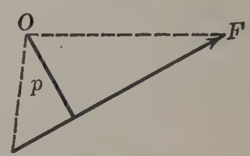

FIG. 43

23. Geometric representation of moments. The moment of the force AB about the point O is $AB \times p$. The area of the triangle ABO is one half of $AB \times p$. Hence the moment of the force AB about the point O is numerically equal to twice the area of the triangle ABO.

From B draw a straight line BN parallel to OA. Draw AC, where C may be taken as any point on BN. Then the moment of the force AC about O is the same as the moment of the force AB about O, since the triangle OAB is equal to the triangle OAC.

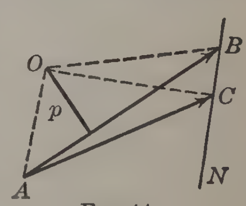

FIG. 44

24. Varignon's theorem. *The sum of the moments of any two coplanar forces about any point in the plane is equal to the moment*

of the resultant of the two forces about the point. Let AB and AD be any two coplanar forces, and let AE be their resultant; also let O be any point in the plane. Draw BC and DF parallel to OA. By § 23, the moment of a force AC about O is the same as the moment of the force AB about O. Likewise the moment of a force AF about O is the same as the moment of the force AD about O. But by similar triangles $AC = FE$ and hence $AC + AF = AE$. Therefore the sum of the moments of AD and AB about O is equal to the moment of AE about O. Hence the theorem follows.

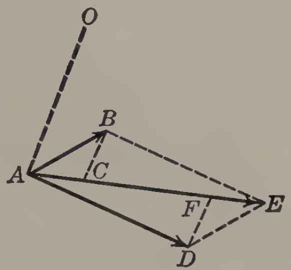

FIG. 45

From the theorem it is evident that the sum of the moments about O of the force AE and any other coplanar force AG is equal to the moment of the resultant of AE and AG about O. Hence the sum of the moments of AB, AD, and AG about O is equal to the moment of the resultant of AB, AD, and AG about O.

In general, *the sum of the moments of any number of coplanar forces about any point in their plane is equal to the moment of the resultant of the forces about the point.*

EXAMPLE

The line of action of a force of 100 lb. makes an angle of 30° with the positive x axis and passes through the point (6, 2). Find the moment of this force about the point (1, 3).

Solutions. 1. The equation of the line of action of the force is

$$y - 2 = \frac{1}{\sqrt{3}}(x - 6).$$

The perpendicular distance from the point (1, 3) to this line is

$$\frac{3\sqrt{3} - 1 + 6 - 2\sqrt{3}}{2} = \frac{5 + \sqrt{3}}{2}.$$

The required moment is $100\left(\frac{5+\sqrt{3}}{2}\right) = 250 + 86.6 = 336.6$ lb.-ft.

2. By Varignon's theorem the moment of the 100-pound force about the point (1, 3) is equal to the sum of the moments of its components about that point.

The horizontal component is 86.6 lb. and its moment is $86.6(1) = 86.6$ lb.-ft. The vertical component is 50 lb. and its moment is $50(5) = 250$ lb.-ft.

Hence the required moment is $250 + 86.6 = 336.6$ lb.-ft.

3. The force of 100 lb. may be applied at any point in its line of action. If the point where $y = 3$ is selected, the horizontal component will have no moment about the point. From the equation of the line, when $y = 3$, $x = 6 + \sqrt{3}$. The arm of the vertical component is therefore $5 + \sqrt{3}$, and the required moment is $(5 + \sqrt{3})50 = 336.6$ lb.-ft.

4. Let an arbitrary point on the line be selected. If $x = -2$ be taken, then

$$y = 2 - \frac{8}{\sqrt{3}}.$$

The arm of the vertical component is 3 and its moment is

$$-(50)(3) = -150 \text{ lb.-ft.}$$

An inspection of Fig. 46 shows that the moment is negative. The arm of the horizontal component is

$$3 - \left(2 - \frac{8}{\sqrt{3}}\right) = \frac{8}{\sqrt{3}} + 1,$$

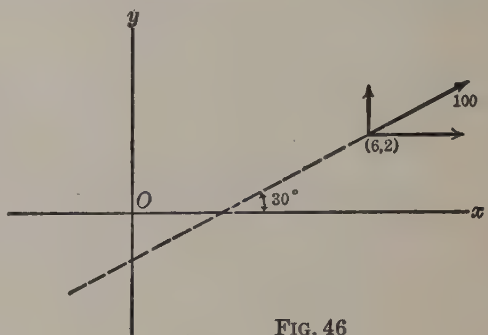

FIG. 46

and its moment is

$$50\sqrt{3}\left(\frac{8}{\sqrt{3}} + 1\right) = 400 + 86.6 = 486.6 \text{ lb.-ft.}$$

The required moment is $486.6 - 150 = 336.6$ lb.-ft.

PROBLEM

A force of 91 lb. acts upward along the line $5y - 12x = 11$, and another force of 80 lb. acts upward along the line $4y - 3x = 22$. Show that their resultant is a force of 165 lb. acting upward along the line $3y - 4x = 13$. Verify the theorem of Varignon by taking moments about the point (8, 3); also about the point (2, − 58).

25. The resultant of parallel forces. The composition or reduction of a system of parallel forces is accomplished by the introduction and later the removal of two equal and opposite collinear forces of arbitrary magnitude.

It is taken as axiomatic that two equal and opposite collinear forces of arbitrary magnitude may be added to or rejected from any system of forces without affecting the resultant of the system. Also it is assumed that the various points at which a system of parallel forces acts are points of a rigid body.

Let P and Q be any two parallel forces acting at the points A and B, respectively, of a rigid body. Join the points of application A and B. At A and B apply two equal and opposite collinear forces F, of arbitrary magnitude. Draw the resultant P' of P and F and similarly the resultant Q' of Q and F, and

produce their lines of action to meet in some point O. Transfer the resultants P' and Q' to act at O and resolve P' and Q' each into its original components P and F, and Q and F, respectively. By the rejection of the two forces F, there remain the two forces P and Q acting at O in a line parallel to their original lines of action. Produce this line to intersect AB in the point C. The resultant of the two parallel forces acting at A and B, respectively, is a single force $P \pm Q$ acting at C in a line parallel to the original forces.

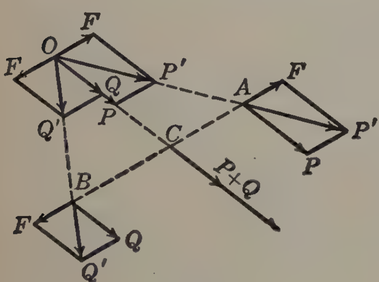

FIG. 47

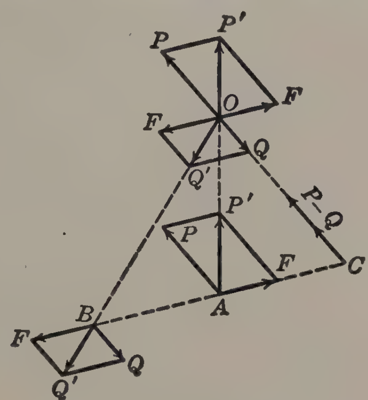

FIG. 48

In Fig. 47 the original forces P and Q act in the same direction, and their resultant is $P + Q$ acting at a point C situated upon the line AB and between A and B.

In Fig. 48 the original forces P and Q act in opposite directions, and their resultant is $P - Q$ acting at a point C situated upon the line AB but external to the segment AB and on the side of the larger force. The position of the point C is determined from similar triangles.

From the triangles OCB and OQQ',

$$\frac{Q}{F} = \frac{OC}{CB}, \tag{1}$$

and from the triangles OCA and OPP',

$$\frac{P}{F} = \frac{OC}{CA}. \tag{2}$$

From (1) and (2),
$$\frac{P}{Q} = \frac{CB}{CA}. \tag{3}$$

Therefore *the resultant of two parallel forces is a parallel force equal to their algebraic sum, and its line of action divides the line*

joining their points of application into segments which are inversely proportional to the forces. The point of division is internal when the forces act in the same direction and external when the forces act in opposite directions; the resultant always lies nearer the larger force. The resultant of *several* parallel forces is obtained by combining the resultant of the first two forces with the third and so on.

An elementary graphical construction for the resultant of two parallel forces serves to exhibit an interesting limiting case.

Let P and Q be any two parallel forces and A and B their points of application respectively. On the line of action of Q make BM equal to P and in the same direction as P. On the line of action of P make AN equal to Q and in the direction opposite to Q. Join M and N and produce, if necessary, to intersect AB at C.

In Figs. 49 and 50, from similar triangles,

$$\frac{BM}{AN} = \frac{CB}{CA} \quad \text{or} \quad \frac{P}{Q} = \frac{CB}{CA}.$$

In Fig. 49 the resultant is $P + Q$ acting in a line through C parallel to the original forces. If $P > Q$, the point C is nearer to A than to B.

In Fig. 50 the resultant is $P - Q$ acting in a line through C parallel to the original forces. If $P > Q$, the point C is to the left of A; if $Q > P$, the point C is to the right of B.

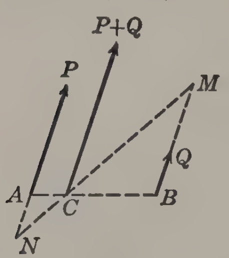

FIG. 49

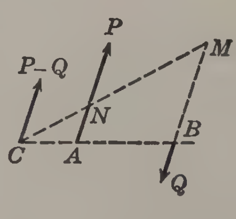

FIG. 50

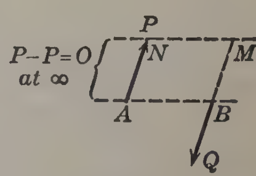

FIG. 51

In Fig. 51, where P and Q are equal and act in opposite directions, the lines MN and AB are parallel and C is at an infinite distance. The resultant is $P - P = 0$. Two equal and oppositely directed forces acting along parallel lines cannot be reduced to a single force at a finite distance. They constitute an irreducible entity, and are called a *couple* (see § 32).

EXAMPLE

Find the resultant of two forces, of 40 lb. and 70 lb., acting in the same direction along parallel lines 6 ft. apart.

Solution. The resultant is a force of $70 + 40 = 110$ lb. Let C be the point where the resultant force intersects any line AB drawn perpendicular to the forces. Also let x be the distance AC. The position of the resultant is found from the equation

$$40\,x = 70(6 - x),$$

$$x = 3.82 \text{ ft.}$$

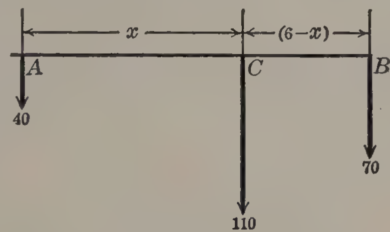

FIG. 52

PROBLEMS

1. Find the resultant of two forces, of 40 lb. and 70 lb., acting in opposite directions along parallel lines 6 ft. apart.
Ans. 30 lb., 8 ft. from the 70-pound force and 14 ft. from the 40-pound force.

2. A wedge 12 in. long is supported in a horizontal position by two vertical cords attached at its ends. The tensions in the cords are 10 lb. and 20 lb. At what point must a single cord be attached to support the wedge in the same position, and what is the tension in the cord?
Ans. 8 in. from A, 30 lb.

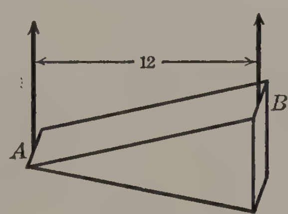

FIG. 53

3. Two oppositely directed parallel forces, of 5 lb. and 12 lb., act on a body. Their resultant is 24 ft. from the smaller force. Find its distance from the greater force. *Ans.* 10 ft.

4. Find the resultant of the force system shown in Fig. 54.
Ans. 8 lb., 2 ft. to the right of the 15-pound force.

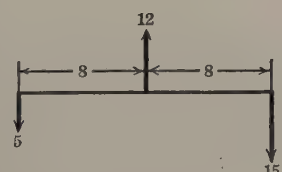

FIG. 54

26. Resolution of a force into two parallel forces. Any given force R acting at a given point C may be replaced by two parallel forces acting at two points, A and B, where A, B, and C are in a straight line.

From § 25 the equations

$$Pa = Qb \tag{1}$$

and

$$P + Q = R \tag{2}$$

must be satisfied, proper attention being given to the algebraic signs of P, Q, a, and b.

If a and b are selected arbitrarily, (1) and (2) determine P and Q.

If P is selected arbitrarily, Q is determined from (2) and then either a or b may be selected arbitrarily and the other determined from (1).

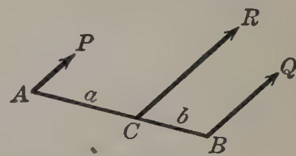

FIG. 55

PROBLEMS

1. Two parallel forces (Fig. 55) act in the same direction at A and B respectively; and their resultant, a force of 15 lb., acts at C. If $AB = 25$ in. and $BC = 6$ in., find the two forces.

Ans. 3.6 lb. and 11.4 lb.

2. Each end of an 18-foot beam of negligible weight is supported on a spring scale. A load of 2400 lb. is placed upon the beam 3 ft. from one end. Find the weight indicated by each scale.

Ans. 2000 lb. and 400 lb.

3. The resultant of the forces P and Q shown in Fig. 56 is 108 lb. Find P and Q.

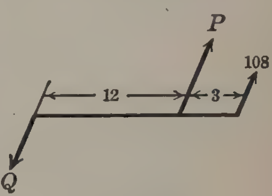

FIG. 56

Ans. $P = 135$ lb., $Q = 27$ lb.

4. The resultant of two parallel forces is 240 lb. One of the forces is 360 lb., opposite in direction to the resultant and 4 ft. from it. Find the other force and its position.

Ans. 600 lb., 2.4 ft. from the resultant.

27. Varignon's theorem for parallel forces. *The sum of the moments of any two parallel forces about any point in the plane of the forces is equal to the moment of the resultant of the two parallel forces about the point.*

Let P and Q be the parallel forces, R their resultant, and let O be the center of moments.

The sum of the moments of P and Q about O is $Pa + Q(a+b+c)$.

The moment of R about O is

$$R(a + b) = (P + Q)(a + b) = Pa + Q(a + b + c),$$

since $Pb = Qc$ by § 25. Hence the theorem follows.

When $P = -Q$ the two parallel forces form a couple (see § 32).

From the theorem it is evident that the sum of the moments about O of the force R and any other parallel coplanar force S is equal to the moment of the resultant of R and S about O. Hence the sum of the moments of P, Q, and S about O is equal to the moment of the resultant of P, Q, and S about O.

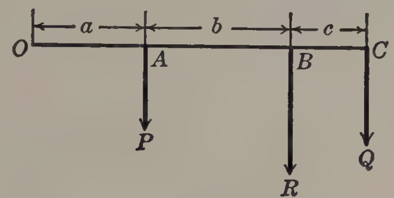

FIG. 57

In general, *the sum of the moments of any number of parallel coplanar forces about any point in their plane is equal to the moment of the resultant of the forces about the point.*

EXAMPLE

Find the resultant of the four coplanar parallel forces shown in Fig. 58.

Solution. The resultant is $20 + 36 - 12 + 24 = 68$ lb. upward. Let H be the point where the line of action of the resultant intersects any line AD, and let $x = AH$. Taking moments about the point A,

$$68\,x = 36(12) - 12(15) + 24(21),$$

from which

$$x = 11.12 \text{ ft.}$$

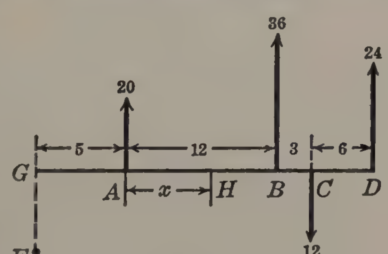

FIG. 58

Hence the resultant is 68 lb., and it acts in a line 0.88 ft. to the left of the 36-pound force.

In order to show that the position of the resultant does not depend upon the selection of the point about which the moments are taken, let the moments be taken about any point E in the plane of the forces.

Let $x = GH$; then

$$68\,x = 20(5) + 36(17) - 12(20) + 24(26),$$

from which $\quad x = 16.12$ ft. from E.

Hence $\quad HB = 0.88$ ft.

28. Analytic expression for the position of a point on the resultant of a system of parallel forces. When the position of the points of application of the several parallel forces are conveniently referred to a system of rectangular axes, the position of a point on the resultant may be obtained from the formulas

of analytic geometry for the coördinates of the point which divides a line joining two points in a given ratio, thus:

$$x_r = \frac{x_1 + kx_2}{1+k}, \qquad y_r = \frac{y_1 + ky_2}{1+k}. \tag{1}$$

Let two parallel forces, P_1 and P_2, act at the points (x_1, y_1) and (x_2, y_2) respectively, and let (x_r, y_r) be a point on the resultant force $P_1 + P_2$. From § 25 the line joining (x_1, y_1) to (x_2, y_2) is divided inversely as the forces; that is, $k = \frac{P_2}{P_1}$. Hence

$$x_r = \frac{x_1 + \left(\frac{P_2}{P_1}\right)x_2}{1 + \frac{P_2}{P_1}}, \qquad y_r = \frac{y_1 + \left(\frac{P_2}{P_1}\right)y_2}{1 + \frac{P_2}{P_1}},$$

or

$$x_r = \frac{P_1x_1 + P_2x_2}{P_1 + P_2}, \qquad y_r = \frac{P_1y_1 + P_2y_2}{P_1 + P_2}.$$

By proceeding in a similar manner, the resultant of $P_1 + P_2$ acting at (x_r, y_r) and P_3 acting at (x_3, y_3) will act at the point given by

$$x = \frac{P_1x_1 + P_2x_2 + P_3x_3}{P_1 + P_2 + P_3}, \quad y = \frac{P_1y_1 + P_2y_2 + P_3y_3}{P_1 + P_2 + P_3}.$$

In general, the coördinates of the point of application of the resultant of any number of parallel forces $P_1, P_2, P_3, \cdots$ acting at $(x_1, y_1, z_1,)$ (x_2, y_2, z_2), (x_3, y_3, z_3), $\cdots$ respectively are given by

$$x_r = \frac{\Sigma(Px)}{\Sigma P}, \quad y_r = \frac{\Sigma(Py)}{\Sigma P}, \quad z_r = \frac{\Sigma(Pz)}{\Sigma P}. \tag{2}$$

The particular case of the resultant of two parallel, oppositely directed, noncollinear forces gives rise to the equation

$$x_r = \frac{P_1x_1 - P_1x_2}{P_1 - P_1} = \frac{P_1(x_1 - x_2)}{0} = \infty,$$

showing that the resultant is a zero force whose point of application is at an infinite distance. This agrees with § 25.

29. Center of parallel forces. From § 28 any number of parallel forces $P_1, P_2, \cdots, P_n$ acting at the points (x_1, y_1, z_1), (x_2, y_2, z_2), $\cdots$, (x_n, y_n, z_n) respectively, and all having direction angles α, β, γ, have a resultant ΣP which has the same direction angles and acts at the point (x_c, y_c, z_c), where

$$x_c = \frac{\Sigma(Px)}{\Sigma P}, \quad y_c = \frac{\Sigma(Py)}{\Sigma P}, \quad z_c = \frac{\Sigma(Pz)}{\Sigma P}.$$

If the points of application of the given parallel forces are fixed and the direction angles α, β, γ allowed to change, the forces all remaining parallel and unaltered in magnitude, then the point of application (x_c, y_c, z_c) of the resultant *remains fixed* and is called *the center of parallel forces.*

If the direction angles α, β, γ remain fixed, but each force assumes a new point of application on its line of action, then the resultant assumes a new point of application on its line of action.

PROBLEMS

1. Solve Example 1, p. 35, (*a*) by taking moments about the point C, (*b*) by taking moments about a point midway between C and D.

2. Find the center of the following forces, all of whose lines of action make equal angles with the coördinate axes:

Force in Pounds	Point of Application
100	(2, 3, 5)
− 60	(2, 1, 0)
110	(0, 0, 0)
200	(− 1, 0, 4)
50	(− 2, − 3, 0)

Ans. (− 0.55, 0.225, 3.25).

30. Center of gravity. The attraction of the earth on every element of any body constitutes a system of parallel forces whose center is called the *center of gravity* of the body. If ΔV represents the element of volume, and γ is the weight per unit volume of any body, the coördinates of the center of gravity of the body, usually designated by $(\bar{x}, \bar{y}, \bar{z})$, are given by the equations

$$\bar{x} = \frac{\Sigma(\gamma \cdot \Delta V \cdot x)}{\Sigma(\gamma \cdot \Delta V)}, \quad \bar{y} = \frac{\Sigma(\gamma \cdot \Delta V \cdot y)}{\Sigma(\gamma \cdot \Delta V)}, \quad \text{and} \quad \bar{z} = \frac{\Sigma(\gamma \cdot \Delta V \cdot z)}{\Sigma(\gamma \cdot \Delta V)}.$$

When the body is homogeneous, the density γ may be canceled. Using the notation of calculus, the equations become

$$\bar{x} = \frac{\int x\,dV}{\int dV}, \quad \bar{y} = \frac{\int y\,dV}{\int dV}, \quad \text{and} \quad \bar{z} = \frac{\int z\,dV}{\int dV}.$$

For a system of bodies whose weights are W_1, W_2, W_3, $\cdots$, and whose centers of gravity are at the points (x_1, y_1, z_1),

(x_2, y_2, z_2), (x_3, y_3, z_3), $\cdots$ respectively, the center of gravity of the system is at the point

$$\bar{x} = \frac{\Sigma(Wx)}{\Sigma W}, \quad \bar{y} = \frac{\Sigma(Wy)}{\Sigma W}, \quad \text{and} \quad \bar{z} = \frac{\Sigma(Wz)}{\Sigma W}.$$

The position of the center of gravity of some simple homogeneous bodies may be inferred from a consideration of their symmetry.

31. Conditions of equilibrium of a system of parallel forces. Let the forces be $P_1, P_2, \cdots, P_n$, acting at points (x_1, y_1, z_1), (x_2, y_2, z_2), $\cdots$, (x_n, y_n, z_n) respectively, and all having direction angles α, β, γ. For equilibrium it is evident that the resultant ΣP must be zero. This is a necessary condition of equilibrium, but it is not sufficient. For suppose all the forces were divided into two groups, and further suppose that the resultant of one group is a force Q; then the resultant of the other group must be $-Q$. Now although ΣP is equal to $Q - Q = 0$, there will not be equilibrium unless the lines of action of Q and $-Q$ coincide.

In order to minimize the algebraic work, let one group consist of the forces $P_1, P_2, \cdots, P_{n-1}$, and the other group of the force P_n alone.

Let (x_k, y_k, z_k) be the point of application of the resultant of $P_1, P_2, \cdots, P_{n-1}$; then

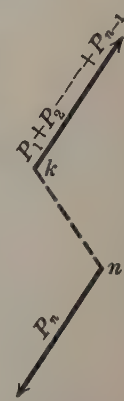

FIG. 59

$$\left.\begin{aligned} x_k &= \frac{P_1x_1 + P_2x_2 + \cdots + P_{n-1}x_{n-1}}{P_1 + P_2 + \cdots + P_{n-1}}, \\ y_k &= \frac{P_1y_1 + P_2y_2 + \cdots + P_{n-1}y_{n-1}}{P_1 + P_2 + \cdots + P_{n-1}}, \\ z_k &= \frac{P_1z_1 + P_2z_2 + \cdots + P_{n-1}z_{n-1}}{P_1 + P_2 + \cdots + P_{n-1}}. \end{aligned}\right\} \quad (1)$$

Also $P_1 + P_2 + \cdots + P_{n-1} = -P_n$, from the condition that $\Sigma P = 0$.

In order that the resultant of $P_1, P_2, \cdots, P_{n-1}$ may lie on the same line as the single force P_n, it is necessary that the line joining the point (x_k, y_k, z_k) and the point (x_n, y_n, z_n) be parallel to all the forces; that is,

$$\frac{x_k - x_n}{\cos\alpha} = \frac{y_k - y_n}{\cos\beta} = \frac{z_k - z_n}{\cos\gamma}. \quad (2)$$

Substituting the values of x_k, y_k, z_k from (1) in (2), replacing $P_1 + P_2 + \cdots + P_{n-1}$ by $- P_n$, and reducing, gives

$$\frac{\Sigma(Px)}{\cos\alpha} = \frac{\Sigma(Py)}{\cos\beta} = \frac{\Sigma(Pz)}{\cos\gamma}.$$

Hence the conditions of equilibrium of a system of parallel forces are

$$\Sigma P = 0 \quad \text{and} \quad \frac{\Sigma(Px)}{\cos\alpha} = \frac{\Sigma(Py)}{\cos\beta} = \frac{\Sigma(Pz)}{\cos\gamma}. \tag{3}$$

For a system of coplanar parallel forces which make an angle α with the x axis the conditions for equilibrium, expressed by (3), become

$$\Sigma P = 0 \quad \text{and} \quad \frac{\Sigma(Px)}{\cos\alpha} = \frac{\Sigma(Py)}{\sin\alpha}. \tag{4}$$

The latter condition, written at length, is

$$P_1x_1 \sin\alpha + P_2x_2 \sin\alpha + \cdots - P_1y_1 \cos\alpha - P_2y_2 \cos\alpha - \cdots = 0,$$

or

$$P_1(x_1 \sin\alpha - y_1 \cos\alpha) + P_2(x_2 \sin\alpha - y_2 \cos\alpha) + \cdots = 0.$$

But $x_1 \sin\alpha - y_1 \cos\alpha = p_1$, $x_2 \sin\alpha - y_2 \cos\alpha = p_2$, $\cdots$,

where $p_1, p_2, \cdots$ are the perpendicular distances from the origin to the lines of action of the forces $P_1, P_2, \cdots$.

Therefore the conditions expressed in (4) may be written

$$\Sigma P = 0 \quad \text{and} \quad \Sigma(Pp) = 0. \tag{5}$$

Moreover, since the origin can be chosen at any point, *the conditions of equilibrium of a system of coplanar parallel forces are*

$$\Sigma P = 0 \quad \text{and} \quad \Sigma(Pp) = 0,$$

or, in words, *the algebraic sum of the forces must be zero, and the sum of the moments of the forces about any point in the plane must be zero.*

PROBLEMS

1. A plank 20 ft. long, weighing 10 lb. per foot, rests horizontally on two supports, one at each end. A man weighing 200 lb. stands on the plank 4 ft. from the left end. Find the reaction at each support. *Ans.* 260 lb., 140 lb.

2. A beam 16 ft. long, weighing 90 lb., is suspended horizontally by two cords attached at its ends. How far from one end must a weight of 180 lb. be hung on the beam in order that the tension in one of the cords may be double the tension in the other? *Ans.* 4 ft.

3. A beam 12 ft. long, weighing 20 lb. per foot, is supported at the left end and at a point 4 ft. from the right end. What is the pressure on each support? *Ans.* 180 lb., 60 lb.

4. What is the pressure on each support of Problem 3 if a weight of 80 lb. at 3 ft. from the left end and a weight of 60 lb. at 2 ft. from the right end are added? *Ans.* 285 lb., 95 lb.

5. A brick is placed on a table with one end projecting 2.5 in. over the edge. If a 3-pound weight is hung on the end of the brick, it is then on the point of tilting over. If an additional weight of 14 lb. is placed on the other end of the brick, it may project 6 in. over the edge. What are the weight and the length of the brick?

Ans. 5 lb., 8 in.

6. A square plate of weight W is supported in a horizontal position by three legs. The two legs at adjacent corners carry one fourth and one fifth of the weight respectively. What are the coördinates of the third leg in terms of the side s? *Ans.* $x = \frac{6}{11}s$, $y = \frac{10}{11}s$.

7. A thin metal plate in the form of an isosceles triangle of height h has its base brazed to a vertical shaft. When the shaft is rotated uniformly, the air resistance on any small area is proportional to the square of its velocity. How far from the shaft is the center of pressure? *Ans.* $\frac{3}{5}h$.

8. Given the three parallel forces P_1 acting at $(a, 0, 0)$, P_2 acting at $(0, b, 0)$, and P_3 acting at $(0, 0, c)$, what must be the relation between the direction cosines of the lines of action of the parallel forces in order that the resultant may pass through the origin?

Ans. $l : m : n = P_1 a : P_2 b : P_3 c$.

9. A circular scale pan of weight 10 oz. and radius 6 in. is suspended by three vertical strings attached to three points at equal distances apart on the circumference of the pan. A weight of 3 oz. may be placed anywhere within or upon a concentric circle of radius 2 in. Find the minimum strength of each string. *Ans.* 5 oz.

10. A body weighs 4 lb. when placed in one pan of a balance, and 4.25 lb. when placed in the other pan. Determine its true weight and the ratio of the lengths of the arms of the balance.

Ans. $W = \sqrt{17}$ lb., $l : l_1 = 4 : \sqrt{17}$.

CHAPTER IV

COUPLES

32. Definitions. A couple is a pair of numerically equal forces which act in opposite directions along parallel noncoincident straight lines. An attempt to reduce these two forces by the usual method of § 25 leads to a resultant equal to zero acting at an infinitely distant point. A couple cannot be resolved into a single force. A couple cannot be replaced by anything simpler, and it is therefore treated as an irreducible entity. The two parallel lines of action of the forces determine a plane, called the *plane of the couple*. The perpendicular distance, p, between the lines of action of the forces is called the *arm of the couple*. The product of either force and the arm is defined as the *moment of the couple*. The moment of the couple in Fig. 60 is Pp. Any line perpendicular to the plane of the couple is called the *axis of the couple*. The positive direction of the axis is related to the sense of rotation of the couple as the advance of a right-handed screw is related to its rotation. A couple is therefore characterized by (1) the magnitude of its moment, (2) the direction of its plane or of its axis in space, and (3) the sense of its rotation. A couple is therefore a vector. The axis indicates the direction of the plane of the couple and also the sense of rotation of the couple. The length of the axis to scale represents the magnitude of the moment of the couple.

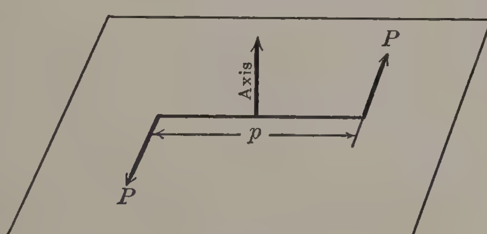

FIG. 60

33. Properties of couples. A *couple* produces or tends to produce rotation of a body. A *force* produces or tends to produce translation of a body. A force may be transferred to any point

of its line of action. A couple has the remarkable property of being transferable to any location whatever, provided its plane maintains its direction. The properties of a couple are presented in the following theorems:

Theorem I. *The sum of the moments of two equal and opposite parallel forces with respect to any point in their plane is constant.*

Let O be any point in the plane of the two forces, and let a be the perpendicular distance from O to one of the forces and $a+p$ the distance to the other force.

Then the sum of the moments of the two forces is

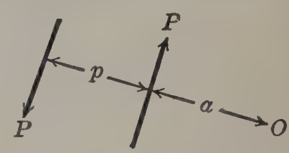

FIG. 61

$$P(a+p) - Pa = Pp = \text{constant}.$$

This constant Pp has been defined as the moment of the couple whose forces are each P and whose arm is p.

Theorem II. *The effect of a couple on a rigid body is not altered if the couple is rotated in its plane through any angle about the mid-point of its arm.*

Let P_1 and P_2 be the equal forces of a couple whose arm is AB. Let CD be the new position of the arm AB. No effect is produced upon the body by the introduction of the equal and opposite forces P_3, P_6 and P_4, P_5, where the forces P_1, P_2, P_3, P_4, P_5, and P_6 have the same magnitude. The forces P_1 and P_6 may be replaced by their resultant K at E. Likewise the forces P_2 and P_5 may be replaced by their resultant K' at F. The forces K and K' annul each other, leaving the couple whose forces are P_3 and P_4 and whose arm is CD. Hence the theorem follows.

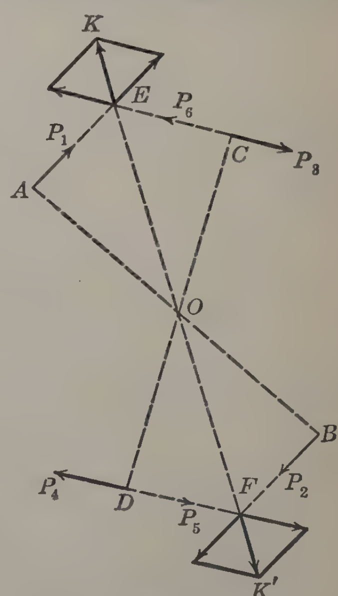

FIG. 62

Theorem III. The effect of a couple on a rigid body is not altered if the couple is moved parallel to itself to any other position in its own plane or in a parallel plane.

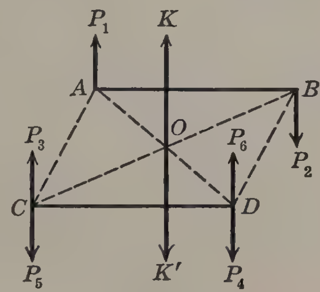

FIG. 63

Let P_1 and P_2 be the equal forces of a couple whose arm is AB. Let CD be the new position of the arm AB. No effect is produced upon the body by the introduction of the equal and opposite forces P_3, P_5 and P_4, P_6, where the forces P_1, P_2, P_3, P_4, P_5, and P_6 have the same magnitude. The forces P_1 and P_6 may be replaced by their resultant K at O. Likewise the forces P_2 and P_5 may be replaced by their resultant K' at O. The forces K and K' annul each other, leaving the couple whose forces are P_3 and P_4 and whose arm is CD. The figure may be viewed as a plane figure or as a figure in three dimensions. Hence the theorem follows.

Theorem IV. The effect of one couple on a rigid body is the same as the effect of another couple of equal moment, their planes being the same, their arms being in the same straight line, and their middle points coincident.

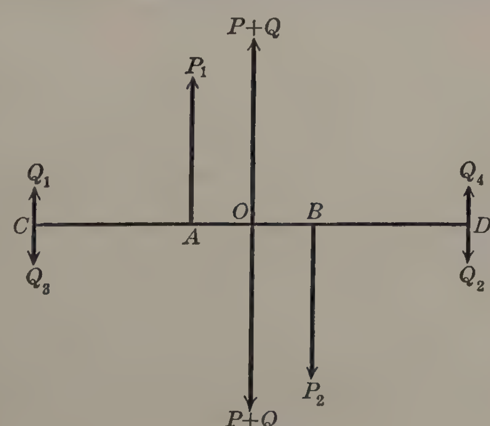

FIG. 64

Let P_1 and P_2 be the equal forces of a couple whose arm is AB. Let Q_1 and Q_2 be the equal forces of another couple whose arm is CD, where

$$P \cdot AB = Q \cdot CD$$

or

$$P \cdot AO = Q \cdot CO.$$

Assuming that the first couple only acts, no effect is produced upon the body by the introduction of the equal and opposite forces Q_1, Q_3 and Q_2, Q_4, where the forces Q_1, Q_2, Q_3, and Q_4 have the same magnitude. The forces P_1

and Q_4 may be replaced by their resultant $P + Q$ acting upward at O (§ 25). Likewise the forces P_2 and Q_3 may be replaced by their resultant $P + Q$ acting downward at O. The forces $P + Q$ and $P + Q$ at O annul each other, leaving the other couple. Hence the theorem follows.

By Theorems II and III *a couple may be transferred from any position in its plane to any other position in its plane or in a parallel plane.*

By Theorem IV either the arm p or the forces P may be changed in magnitude in any way whatever, provided the moment Pp remains unchanged.

34. Resultant of any number of couples. CASE I. *Couples in the same or in parallel planes.* The resultant of any number of couples in the same or parallel planes is a couple lying in the same or any parallel plane whose moment is the algebraic sum of the moments of the several couples.

CASE II. *Couples in planes inclined to each other.* The resultant of several couples lying in planes inclined to each other is a couple whose vector is the vector sum of the vectors of the several couples. In other words, if the several couples are represented by their vectors (axes), they may be combined or resolved as if they were forces acting at a point.

Another method of finding the resultant of two couples lying in different planes is as follows:

Let the moment of the couple lying in the plane A be Pp, and the moment of the couple lying in the plane B be Qq.

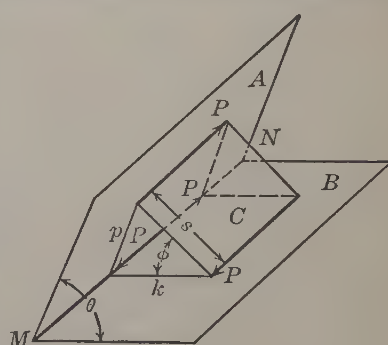

FIG. 65

Move the couple Pp in its plane so that one of its forces P coincides with MN, the line of intersection of the two planes A and B. Change the couple Qq to a couple whose forces are each P; that is, determine k so that $Qq = Pk$. Move the couple Pk in its plane B so that one of its forces P lies in MN but in the opposite direction from the force P of the couple Pp. The two forces P and P lying in MN annul each other, leaving a resultant couple

whose moment is Ps, lying in the plane C. The arm s of the new couple lying in the plane C is determined from the equation

$$s^2 = p^2 + k^2 - 2\,pk \cos \theta,$$

where θ is the angle between the planes A and B. The angle ϕ which the new plane C makes with the plane B is determined from the relation

$$\frac{\sin \phi}{p} = \frac{\sin \theta}{s}.$$

The vectors representing these couples and the process of combining them are shown in Fig. 66, which is a section of Fig. 65, made by a plane perpendicular to the line MN. In Fig. 66 one force of the couple Pp acts perpendicular to the plane of the paper upward at o; the other force acts downward at a. Hence the vector for the couple Pp is a line oc of length Pp to scale and perpendicular to oa. Similarly the vector for the couple Pk is a line od of length Pk to scale and perpendicular to ob. The resultant of the vectors oc and od is a vector oe.

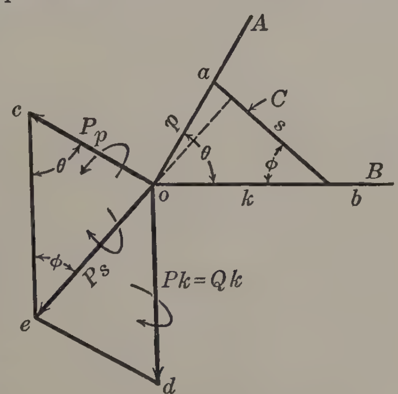

FIG. 66

Since each of the forces in all three couples is P,

$$Pp : Pk : Ps = p : k : s.$$

Also, since the triangles oab and oce have an angle θ common and the adjacent sides proportional, they are similar, and hence

$$oc : ce : oe = p : k : s.$$

Therefore the vector oe represents the couple of moment Ps.

Evidently the resultant of these two couples can be combined with a third couple, and so on.

Conversely, any couple can be resolved into any number of couples by representing the given couple by a vector and resolving the vector into any components in exactly the same manner as a force is resolved into components.

It is evident that Case I is included under Case II by making $\theta = 0$.

35. A force and a couple replaced by a parallel force. Any force and a couple lying in the same plane and acting upon a rigid body may be replaced by a single force equal to the original force and parallel to it. Let P be the given force. Also let F be the equal forces and f the arm of the given couple. Reduce the moment Ff of the couple to Pk, where k is obtained from the identity $Pk = Ff$. Move the couple Pk in its plane so that one of the forces annuls the original force P, leaving the single force P parallel to the original force P and at a perpendicular distance $\frac{Ff}{P} = k$ from it.

It should be noticed that although a couple and a force lying in the same plane can be reduced to a single force, a *couple alone* cannot be further reduced.

36. The parallel displacement of a force by the introduction of a couple. A force P acting at any point A of a rigid body may be transferred parallel to itself to act at any point B provided a couple is introduced whose moment is Pp, where p is the distance between the lines of action of the original and transferred forces.

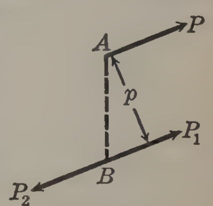

FIG. 67

Let P be the given force acting at the point A of the rigid body. No effect is produced upon the rigid body by the introduction of the equal and opposite forces P_1 and P_2, each equal to P, at B.

The forces P and P_2 form a couple whose moment is Pp. Hence the force P at A is equivalent to the force $P_1 = P$ at B and the couple whose moment is Pp.

PROBLEMS

1. $ABCD$ is a square whose side is 2 ft. long. Along AB, BC, CD, and DA act forces of 2 lb., 4 lb., 16 lb., and 2 lb. respectively, and along AC and DB act forces of $6\sqrt{2}$ lb. and $8\sqrt{2}$ lb. Show that the forces are equivalent to a couple of -24 lb.-ft.

2. Given a force of 60 lb. acting upward along the line $x = 5$, a force of 10 lb. acting upward along the line $x = 3$, a force of 20 lb. acting downward along the line $x = 1$, and a force of 30 lb. acting upward along the y axis; determine the magnitude and line of action of a force which, combined with the given forces, forms a couple whose moment is 240 lb.-units.

Ans. 80 lb. downward along $x = 0.875$.

3. The sides of any plane polygon, traced in order, represent forces to scale. Show that their resultant is a couple whose moment is represented by twice the area of the polygon.

4. The system of forces acting on the beam shown in Fig. 68 is in equilibrium. Replace the couple and force by a single force and show that the weight of the beam is 10 lb.

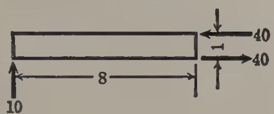

FIG. 68

5. A force of P lb. acts along one side of a square. Replace the force by three forces which act along the other three sides of the square respectively.

6. A force of 10 lb. acts to the left along the line $y = 1$, a force of 30 lb. acts to the left along the line $y = 2$, and a force of 10 lb. acts to the right along the line $y = 4$. What forces acting along the lines $y = -2$ and $y = 5$ will hold the three given forces in equilibrium?

Ans. 17.14 lb. and 12.86 lb.

CHAPTER V

CENTER OF GRAVITY

37. Center of gravity. Gravitation is defined as the mutual action between masses of matter by virtue of which every elementary mass tends toward every other elementary mass with a force varying directly as the product of the masses and inversely as the square of their distance apart. This force or action of gravity between a body and the earth, called the weight of the body, is always present in every problem in mechanics.

For small differences of distance from the center of the earth, the weight of a body is practically constant; and therefore if γ is the weight per unit volume, the weight of an elementary volume is $\gamma \cdot \Delta V$. Since the bodies considered are small in comparison with the earth, the lines of action of the force of gravity upon elementary volumes of the bodies may be considered parallel, and therefore these forces $\gamma \cdot \Delta V$ constitute a system of parallel forces. From § 28 the resultant of a system of parallel forces $P_1, P_2, \cdots$ acting at fixed points (x_1, y_1, z_1), $(x_2, y_2, z_2), \cdots$ is a force ΣP which acts at a *fixed* point (x_c, y_c, z_c) called the center of parallel forces, where

$$x_c = \frac{\Sigma(Px)}{\Sigma P}, \quad y_c = \frac{\Sigma(Py)}{\Sigma P} \quad \text{and} \quad z_c = \frac{\Sigma(Pz)}{\Sigma P}. \tag{1}$$

The position of this center of parallel forces is independent of the direction cosines of the parallel lines along which the forces act and depends only on the relative magnitudes and the points of application of the parallel forces.

When the parallel forces $P_1, P_2, \cdots$ are replaced by the weights of the elementary volumes, $\gamma \cdot \Delta V$, of a body, the center of parallel forces is called the *center of gravity*, and the resultant of the parallel forces becomes the weight of the body. It is clear that the center of gravity of a body is a fixed point through which the line of action of its weight passes, no matter in what position the body may be placed.

The coördinates of the center of gravity are given by the equations

$$\bar{x}=\frac{\Sigma(\gamma\cdot\Delta V\cdot x)}{\Sigma(\gamma\cdot\Delta V)},\quad \bar{y}=\frac{\Sigma(\gamma\cdot\Delta V\cdot y)}{\Sigma(\gamma\cdot\Delta V)},\quad \bar{z}=\frac{\Sigma(\gamma\cdot\Delta V\cdot z)}{\Sigma(\gamma\cdot\Delta V)}. \tag{2}$$

When the body is homogeneous, the weight per unit volume, γ, may be canceled, and the coördinates of the center of gravity become

$$\bar{x}=\frac{\Sigma(\Delta V\cdot x)}{\Sigma(\Delta V)},\quad \bar{y}=\frac{\Sigma(\Delta V\cdot y)}{\Sigma(\Delta V)},\quad \bar{z}=\frac{\Sigma(\Delta V\cdot z)}{\Sigma(\Delta V)}. \tag{3}$$

In this form it is clear that the idea of weight has disappeared from the formulas, and therefore they may be considered as formulas for the *center of volume.*

In most problems it is necessary to make the elementary volume infinitesimal, and hence the formulas become

$$\bar{x}=\frac{\int x\,dV}{\int dV},\quad \bar{y}=\frac{\int y\,dV}{\int dV},\quad \bar{z}=\frac{\int z\,dV}{\int dV}. \tag{4}$$

A curved bar of uniform cross section small in comparison with its length may be considered as a curved line. A thin curved sheet of metal may be considered as a surface. Thus there arises the problem of finding the center of gravity of a line or a surface as well as of a solid.

Evidently the formulas for the center of gravity of a surface are

$$\bar{x}=\frac{\int x\,dA}{\int dA},\quad \bar{y}=\frac{\int y\,dA}{\int dA},\quad \bar{z}=\frac{\int z\,dA}{\int dA}, \tag{5}$$

and the formulas for the center of gravity of a line are

$$\bar{x}=\frac{\int x\,ds}{\int ds},\quad \bar{y}=\frac{\int y\,ds}{\int ds},\quad \bar{z}=\frac{\int z\,ds}{\int ds}. \tag{6}$$

38. Center of gravity of a system of bodies or of a compound body. The center of gravity of a system of bodies or of a compound body which consists of parts whose weights and centers

of gravity are known is obtained by the use of the formulas for the center of parallel forces.

Let W_1, W_2, $\cdots$ be the weights and (x_1, y_1, z_1), (x_2, y_2, z_2), $\cdots$ the coördinates of the centers of gravity of the bodies; then the coördinates of the center of gravity of the system of bodies are

$$\bar{x} = \frac{\Sigma(Wx)}{\Sigma W}, \quad \bar{y} = \frac{\Sigma(Wy)}{\Sigma W}, \quad \bar{z} = \frac{\Sigma(Wz)}{\Sigma W}. \tag{1}$$

If (1) is cleared of fractions,

$$\bar{x} \cdot \Sigma W = \Sigma(Wx), \quad \bar{y} \cdot \Sigma W = \Sigma(Wy), \quad \bar{z} \cdot \Sigma W = \Sigma(Wz). \tag{2}$$

Since there are no restrictions on the direction of the weights, they may be taken parallel to the y axis, making the xz plane horizontal. The first equation of (2) may then be interpreted as follows: the moment of the resultant of the weights about the z axis is equal to the sum of the moments of the weights. The other two equations may be similarly interpreted.

The six sets of equations in § 37 may be given analogous interpretations.

39. Center of gravity determined from symmetry. The positions of the center of gravity of some homogeneous bodies are evident by inspection. Certain inferences may be made about the location of the center of gravity of some bodies from considerations of symmetry. The center of gravity of a body which is symmetrical with respect to a plane, line, or point usually lies in the plane, line, or point. The center of gravity of a straight line is evidently the middle point of the line. The center of gravity of a sphere is at its center. The center of gravity of a right circular cone or cylinder lies on its axis. The center of gravity of a parallelogram is at the intersection of its diagonals. The center of gravity of two equal weights is the mid-point of the line joining their centers of gravity, and so on.

40. General rule for finding the center of gravity. Divide the body into portions, finite or infinitesimal, such that the weight and the coördinates of each portion are known.

Multiply the weight of every portion by the coördinate of its center of gravity and add or integrate the products.

This sum or integral divided by the total weight gives the coördinate of the center of gravity of the body.

When the body, surface, or line is homogeneous, the word "weight" in the above rule may be replaced by "volume," "area," or "length," respectively, without affecting the validity of the general statement in any way.

41. Equivalent particles. Frequently the actual body whose center of gravity is required may be replaced by a system of particles whose combined weight is the same as the body and which are so situated as to have the same center of gravity as the body. For example, consider a triangular plate. Assume the triangular plate to be cut into small strips by lines parallel to one side; the center of gravity of each strip lies at its middle point. The locus of these middle points is a median, and hence the center of gravity of the triangular plate lies on a median and therefore at the intersection of the medians. If three particles, each one third the weight of the triangular plate, are placed at the vertices of the triangle, the center of gravity of any two of them is midway between them. If these two weights are placed at this middle point, the center of gravity of the three weights, now located in only two positions, will lie on the median and divide it in the ratio of 1 to 2. Hence it is clear that the center of gravity of all three particles as originally placed is at the intersection of the medians of the triangle.

42. Center of gravity of lines. The center of gravity of any line or system of lines may be found by considering the lines as bodies of very small constant cross section and of uniform density.

EXAMPLES

1. Find the center of gravity of the perimeter (not the area) of the triangle ABC shown in Fig. 69.

Solution. In accordance with the general rule the perimeter is divided into the sides a, b, and c whose centers of gravity are at the midpoints of the sides respectively. The sum of the moments of the sides about the x axis is

$$a\left(\frac{a \sin C}{2}\right) + c\left(\frac{a \sin C}{2}\right) + b(0).$$

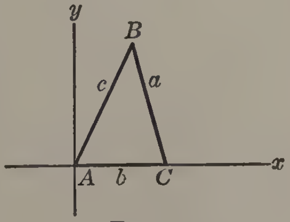

FIG. 69

Therefore $\bar{y} = \dfrac{a \sin C(a + c)}{2(a + b + c)}.$

Similarly, $\bar{x} = \dfrac{b^2 + c^2 \cos A + a^2 \cos C + 2\, ac \cos A}{2(a + b + c)}.$

2. Find the center of gravity of an arc of a circle.

Solution. Let the x axis coincide with the bisector of the arc whose central angle is $2\,\alpha$. Also let the origin be taken at the center of the circle. An element of the arc is $r \cdot d\theta$, and its coördinates are (x, y) or $(r\cos\theta, r\sin\theta)$.

Therefore

$$\bar{x} = \frac{\int_{-\alpha}^{+\alpha} r \cdot d\theta \cdot r\cos\theta}{\int_{-\alpha}^{+\alpha} r \cdot d\theta} = \frac{r\sin\alpha}{\alpha}.$$

From symmetry, $\bar{y} = 0.$

For a semicircular arc, $\bar{x} = \dfrac{2\,r}{\pi}.$

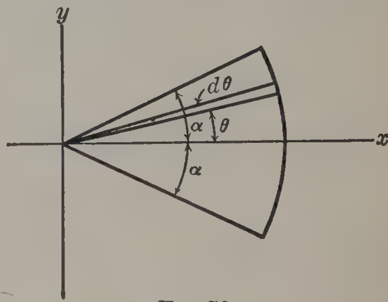

FIG. 70

3. Find the position of the center of gravity of the lines AB and AC shown in Fig. 71.

Solution. The center of gravity of the line AB is at its mid-point D. Also the center of gravity of AC is at its mid-point E. Therefore the center of gravity of the two lines is the point G on the line joining D and E, where

$$\frac{DG}{GE} = \frac{AC}{AB}.$$

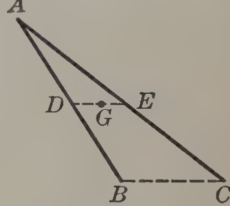

FIG. 71

It is important to notice that G does not lie upon the bisector of the angle A; neither does it lie on the median drawn from A, unless $AB = AC$.

PROBLEMS

1. A uniform rod 6 ft. long, weighing 9 lb., carries weights of 1 lb., 2 lb., 3 lb., 4 lb., 5 lb., and 6 lb. at the 1-foot, 2-foot, 3-foot, 4-foot, 5-foot, and 6-foot marks from one end. Find the center of gravity.

Ans. 3.93 ft. from the end.

2. Find the coördinates of the center of gravity of the light uniform wire shown in Fig. 72.

Ans. $\bar{x} = 0,\ \bar{y} = \dfrac{2\,r}{\pi}.$

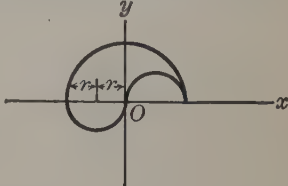

FIG. 72

3. A slender bar is bent sharply to form two sides of a triangle whose lengths are m and n. Show that the center of gravity of the bent bar lies on a line which divides the angle between m and n into two angles, M and N, such that $\dfrac{\sin M}{\sin N} = \dfrac{n^2}{m^2}.$

4. Find the center of gravity of the framework shown in Fig. 73. *Ans.* $\bar{x} = 0$,

$$\bar{y} = \left(\frac{21 - 6\sqrt{3}}{37}\right)h.$$

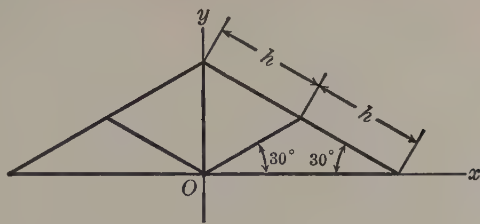

FIG. 73

5. Three axles, bearing loads of 24 T., 20 T., and 16 T., are fixed at distances of 8 ft. and 12 ft. by a frame, as shown in Fig. 74. To what position must the wheels be moved on the 48-foot girder in order that the reactions at A and B may be in the ratio of 13 to 23? The girder weighs 12 T.

Ans. $x = 24$ ft.

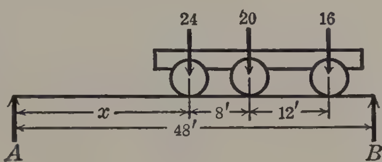

FIG. 74

6. Find the coördinates of the center of gravity of the bodies A, B, and C as shown in Fig. 75. The body A is in the yz plane, the body B is in the xz plane, and the body C is in a plane making an angle of 60° with the xz plane. *Ans.* (2.51, 2.79, 4.55).

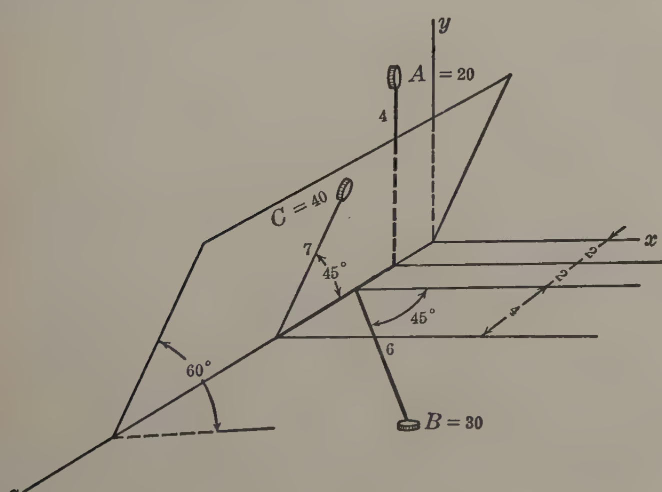

FIG. 75

7. Three uniform rods are rigidly united so as to form half of a regular hexagon. If it is suspended from one of the angular points, show that one rod will be horizontal.

43. Center of gravity of plane areas. The center of gravity of any plane area may be found by considering the plane area as a body of very small constant thickness and of uniform density.

EXAMPLES

1. Find the center of gravity of the area of the triangle shown in Fig. 76.

Solution. Let b be the base and h the altitude of any plane triangular area. Let y be the distance from the base to any elementary strip of width dy and length x, parallel to the base.

Then the area of the strip is $x \cdot dy$, and the distance of its center of gravity from the base is y. Therefore the distance of the center of gravity of the triangular area from its base is

$$\bar{y} = \frac{\int_0^h x\,dy \cdot y}{\int_0^h x\,dy}.$$

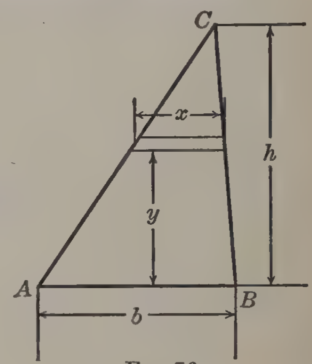

FIG. 76

Before the integration can be carried out, x must be expressed in terms of y. From similar triangles,

$$\frac{x}{b} = \frac{h-y}{h}.$$

Hence $$\bar{y} = \frac{\frac{b}{h}\int_0^h (h-y)\,y\,dy}{\frac{b}{h}\int_0^h (h-y)\,dy} = \frac{h}{3}.$$

The center of gravity therefore lies in a line parallel to the base and at a distance of $\frac{1}{3}h$ from the base. By § 41 the center of gravity lies on the median and hence at the intersection of the medians.

2. Find the center of gravity of the sector of a circle shown in Fig. 77.

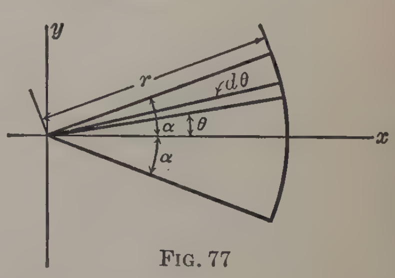

FIG. 77

Solution. Let the x axis coincide with the bisector of the arc whose central angle is 2α. The area of an element is $\frac{r^2\,d\theta}{2}$, and the distance of its center of gravity from the y axis is $\frac{2}{3} r \cos\theta$.

Hence $$\bar{x} = \frac{\int_{-\alpha}^{+\alpha} \frac{1}{2} r^2\,d\theta \cdot \frac{2}{3} r\cos\theta}{\int_{-\alpha}^{+\alpha} \frac{1}{2} r^2\,d\theta} = \frac{\frac{r^3}{3}\int_{-\alpha}^{+\alpha} \cos\theta\,d\theta}{\frac{r^2}{2}\int_{-\alpha}^{+\alpha} d\theta} = \frac{2}{3}\,\frac{r\sin\alpha}{\alpha}.$$

This result may be obtained more simply. The locus of the center of gravity of the elementary areas is a concentric arc of radius $\frac{2}{3}r$. By § 42, Example 2, the center of gravity of any circular arc is $\frac{r \sin \alpha}{\alpha}$.

Hence $$\bar{x} = \frac{2}{3}\frac{r \sin \alpha}{\alpha}.$$

From symmetry, $$\bar{y} = 0.$$

For the area of a semicircle, $\bar{x} = \frac{4r}{3\pi}$, $\bar{y} = 0$.

3. Find the center of gravity of any quadrilateral area.

Solution. Any quadrilateral area may be divided into two triangles whose areas and centers of gravity are known, and hence the center of gravity of the quadrilateral may be found. Any plane polygon may be treated in a similar manner.

In § 41 it was shown that the center of gravity of three particles, each one third the weight of a triangular area, placed at the vertices have the same center of gravity as the triangular area. It is not difficult to show that the equivalent particles for a quadrilateral area consist of four particles, each *one third* the weight of the quadrilateral area placed at the vertices, together with a fifth equal negative particle placed at the intersection of the diagonals. (See Routh's "Analytical Statics," Vol. I, p. 258.)

4. Find the center of gravity of any trapezoid.

Solution. Let $ABCD$, Fig. 78, be any trapezoid whose upper base is b, lower base a, and whose altitude is h. Also let O be the intersection of the diagonals and G the center of gravity.

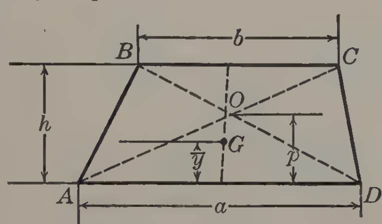

FIG. 78

Evidently the center of gravity lies on the median. The ordinate $\bar{y}$ may be determined by several methods:

1. *Equivalent particles.* Equating the moments of the equivalent particles about AD to the moment of the area gives

$$\frac{Ah}{3} + \frac{Ah}{3} - \frac{Ap}{3} = A\bar{y}.$$

From the similar triangles AOD and BOC,

$$p = \frac{ah}{a+b}.$$

Therefore $$\frac{2\,Ah}{3} - \frac{Aah}{3(a+b)} = A\bar{y}.$$

Hence $$\bar{y} = \frac{h}{3}\left(\frac{a+2\,b}{a+b}\right).$$

2. *Two triangles.* Equating the moments of the areas of the two triangles ABC and ACD to the moment of the trapezoid about the line AC gives

$$\frac{bh}{2}\left(\frac{2}{3}h\right) + \frac{ah}{2}\left(\frac{h}{3}\right) = \frac{(a+b)h}{2}(\bar{y}).$$

Therefore $$\bar{y} = \frac{h}{3}\left(\frac{a+2\,b}{a+b}\right).$$

3. *Integration.* Divide the area into elementary strips parallel to AD, whose lengths are x and width dy. The distance of the center of gravity of any strip from AD is y.

Hence $$\bar{y} = \frac{\int_0^h x\,dy \cdot y}{\int_0^h x\,dy}.$$

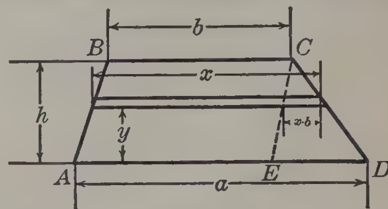

FIG. 79

In order to express x in terms of y, draw CE parallel to BA. From similar triangles,

$$\frac{x-b}{h-y} = \frac{a-b}{h},$$

from which $$x = \frac{ah - ay + by}{h}.$$

Therefore $$\bar{y} = \frac{\int_0^h (ah - ay + by)y\,dy}{\int_0^h (ah - ay + by)dy} = \frac{h}{3}\left(\frac{a + 2\,b}{a + b}\right).$$

4. *Graphical method.* The center of gravity lies on the median. It lies also on a line EF, which is determined as follows: Produce one of the parallel sides BC in either direction a distance equal to the other parallel side; that is, make $CF = AD$. Similarly, produce AD in the opposite direction a distance equal to the first side; that is, make $AE = BC$. The center of gravity of the quadrilateral lies at G, the intersection of the median and the line EF. The proof of this construction is left to the student.

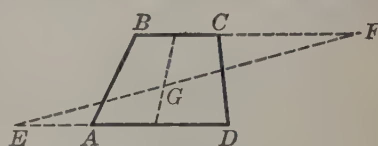

FIG. 80

5. Find the center of gravity of the sector of a flat ring.

Solution. Let the area $ABCD$, Fig. 81, which is bounded by two concentric arcs and two radii, be referred to axes one of which bisects the central angle, the origin being at the center. The moment of the area OBC about the y axis is equal to the sum of the moments of the area OAD and the area $ABCD$.

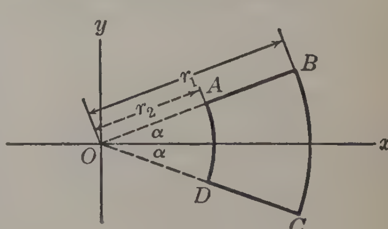

FIG. 81

Therefore

$$r_1^2\alpha\left(\frac{2}{3}\,\frac{r_1 \sin\alpha}{\alpha}\right) = r_2^2\alpha\left(\frac{2}{3}\,\frac{r_2 \sin\alpha}{\alpha}\right) + (r_1^2\alpha - r_2^2\alpha)\bar{x}.$$

Hence $$\bar{x} = \frac{2}{3}\left(\frac{\sin\alpha}{\alpha}\right)\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right).$$

When $\alpha = 90°$. $$\bar{x} = \frac{4}{3\,\pi}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right).$$

6. Find the centers of gravity of the areas shown in Fig. 82 and Fig. 83.

Solution. The position of the center of gravity of an area composed of rectangles, triangles, segments, and sectors of circles, etc. may be found by taking moments.

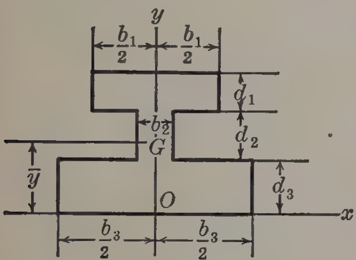

FIG. 82

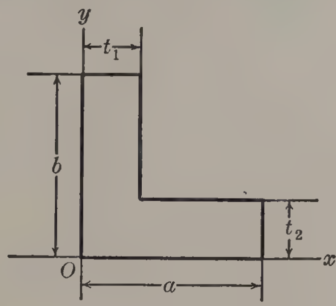

FIG. 83

For Fig. 82

$$\bar{y} = \frac{b_1 d_1\left(\frac{d_1}{2} + d_2 + d_3\right) + b_2 d_2\left(\frac{d_2}{2} + d_3\right) + b_3 d_3\left(\frac{d_3}{2}\right)}{b_1 d_1 + b_2 d_2 + b_3 d_3}.$$

For Fig. 83

$$\bar{y} = \frac{t_1(b - t_2)\left(\frac{b + t_2}{2}\right) + a t_2\left(\frac{t_2}{2}\right)}{t_1(b - t_2) + a t_2},$$

$$\bar{x} = \frac{a t_2\left(\frac{a}{2}\right) + (b - t_2)t_1\left(\frac{t_1}{2}\right)}{b t_1 + t_2(a - t_1)}.$$

7. Find the coördinates of the center of gravity of an area bounded by the parabola $y^2 = 2\,px$, the x axis, and the line $x = a$.

Solution. The element of area is $y\,dx$, and the coördinates of its center of gravity are $\left(x, \frac{y}{2}\right)$.

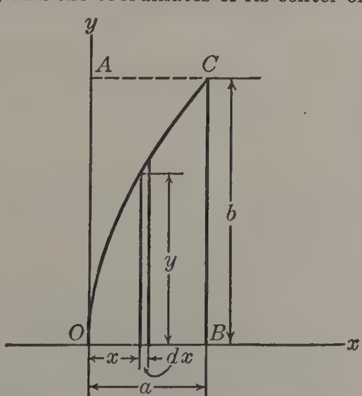

FIG. 84

Therefore

$$\bar{x} = \frac{\int_0^a y\,dx \cdot x}{\int_0^a y\,dx} = \frac{\sqrt{2\,p}\int_0^a x^{\frac{3}{2}}\,dx}{\sqrt{2\,p}\int_0^a x^{\frac{1}{2}}\,dx} = \frac{3\,a}{5}$$

and

$$\bar{y} = \frac{\int_0^a y\,dx \cdot \frac{y}{2}}{\int_0^a y\,dx} = \frac{p\int_0^a x\,dx}{\sqrt{2\,p}\int_0^a x^{\frac{1}{2}}\,dx}$$

$$= \frac{3\sqrt{2\,ap}}{8} = \frac{3\,b}{8}.$$

8. Find the center of gravity of the portion of an ellipse bounded by the lines $x = k$, $x = h$, and the x axis.

Solution. The elementary area is $y\,dx$, and its center of gravity is the point $\left(x, \frac{y}{2}\right)$.

Therefore

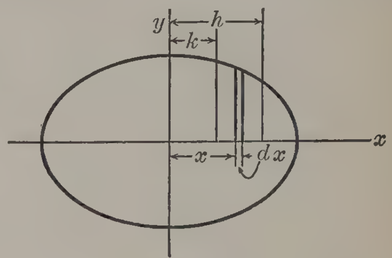

FIG. 85

$$\bar{x} = \frac{\int_k^h xy\,dx}{\int_k^h y\,dx} = \frac{\int_k^h x\sqrt{a^2 - x^2}\,dx}{\int_k^h \sqrt{a^2 - x^2}\,dx}$$

$$= \frac{\frac{2}{3}\left[(a^2 - k^2)^{\frac{3}{2}} - (a^2 - h^2)^{\frac{3}{2}}\right]}{h\sqrt{a^2 - h^2} - k\sqrt{a^2 - k^2} + a^2\left[\arcsin\frac{h}{a} - \arcsin\frac{k}{a}\right]}.$$

Similarly,

$$\bar{y} = \frac{\frac{b}{a}\left[a^2(h - k) - \frac{1}{3}(h^3 - k^3)\right]}{h\sqrt{a^2 - h^2} - k\sqrt{a^2 - k^2} + a^2\left[\arcsin\frac{h}{a} - \arcsin\frac{k}{a}\right]}.$$

For a quadrant of an ellipse

$$\bar{x} = \frac{4\,a}{3\,\pi}, \qquad \bar{y} = \frac{4\,b}{3\,\pi}.$$

PROBLEMS

1. Find the coördinates of the center of gravity of the area $ABCO$ in Fig. 86. *Ans.* $\bar{x} = \frac{7}{18}\,s$, $\bar{y} = \frac{4}{9}\,s$.

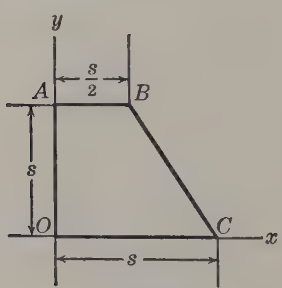

FIG. 86

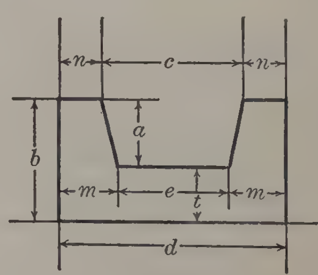

FIG. 87

2. Find the coördinates of the center of gravity of the channel section shown in Fig. 87.

$$Ans.\ \bar{x} = \frac{d}{2},\ \bar{y} = \frac{b^2 n + \frac{ct^2}{2} + \frac{a(m - n)}{3}(b + 2\,t)}{A}.$$

3. Determine the coördinates of the center of gravity of the built-up section shown in Fig. 88.

Ans. 9.836 in.

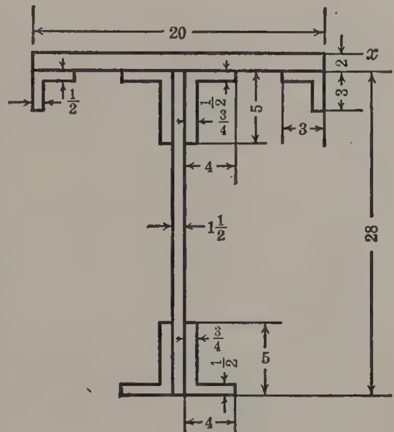

FIG. 88

4. Determine the ordinate $\bar{y}$ of the point G, so that G is the center of gravity of the hatched area in Fig. 89.

$$Ans.\ \bar{y} = \frac{(3-\sqrt{3})a}{2}.$$

5. The profile of a crank consists of two semicircular ends of 8-inch radius and 12-inch radius respectively, centered at points 3 ft. apart and joined by straight lines. The crank is of uniform thickness and is pierced by holes of 3-inch radius and 5-inch radius, as shown in Fig. 90. How far is the center of gravity from the center of the larger hole?

Ans. Area of section $= 939.91$ sq. in., $\bar{y} = 14.774$ in.

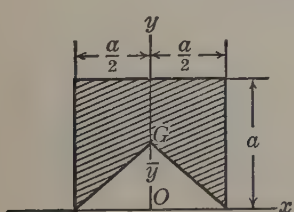

FIG. 89

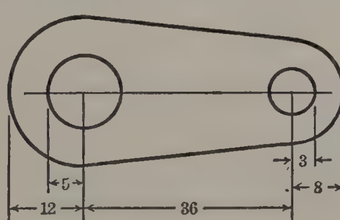

FIG. 90

6. Show that the center of gravity of any quadrilateral area coincides with the point of intersection of the diagonals of a parallelogram whose sides are determined by the third points of the quadrilateral.

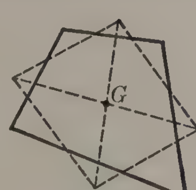

FIG. 91

7. Find the center of gravity of the area in the first quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$.

$$Ans.\ \bar{x} = \frac{2a}{3(\pi - 2)},\ \bar{y} = \frac{2b}{3(\pi - 2)}.$$

44. Center of gravity of surfaces. The center of gravity of any surface may be found by considering the surface as a body of very small constant thickness and of uniform density.

EXAMPLES

1. Find the center of gravity of the surface of a pyramid or cone. (Base not included.)

Solution. The center of gravity of each triangular face is one third the distance from the mid-point of its base to the vertex of the pyramidal surface. Therefore, by similar triangles, they are each at a distance $\frac{h}{3}$ from the base, where h is the altitude. Hence the center of gravity of the pyramidal surface lies in a plane parallel to the base and at a distance $\frac{h}{3}$ above it.

The position of the center of gravity G in this plane coincides with the center of gravity of the areas of the faces placed at their centers of gravity.

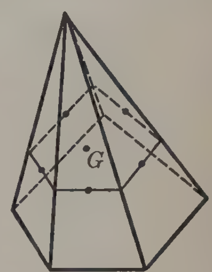

FIG. 92

The center of gravity G does not lie on the line joining the center of gravity of the *perimeter* of the base with the vertex, nor upon the line joining the center of gravity of the *area* of the base with the vertex except in case the pyramid is a *regular* pyramid.

Since the portion of the conical surface of a cone cut off by a plane may be considered as the limit of a pyramidal surface the bases of whose triangular faces approach zero, it follows that its center of gravity lies in a plane parallel to the base of the cone at a distance $\frac{h}{3}$ above it.

The center of gravity of the surface of a *right* circular cone is two thirds of the way from the vertex to the center of the base. In case of an oblique circular cone or an elliptic cone, the center of gravity of the conical surface does not lie on the line joining the vertex to the center of the base. (See Routh's "Analytical Statics," Vol. I, Art. 419.)

2. Find the center of gravity of a spherical surface included between two parallel planes.

Solution. From symmetry, $z = 0$ and $\bar{y} = 0$.

The elemental area is $r\,d\theta \cdot 2\,\pi r \sin\theta$, and the x coördinate of its center of gravity is $r\cos\theta$. Hence

$$\bar{x} = \frac{\int_{\theta_1}^{\theta_2} 2\,\pi r^3 \sin\theta\cos\theta\,d\theta}{\int_{\theta_1}^{\theta_2} 2\,\pi r^2 \sin\theta\,d\theta}$$

$$= \frac{r}{2}\left(\frac{\cos^2\theta_2 - \cos^2\theta_1}{\cos\theta_2 - \cos\theta_1}\right) = \frac{r}{2}(\cos\theta_2 + \cos\theta_1).$$

Therefore the center of gravity is halfway between the two planes.

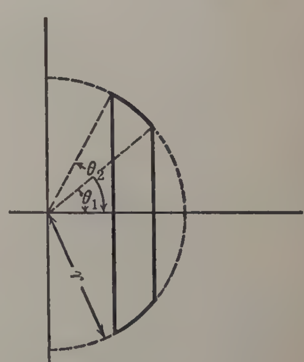

FIG. 93

PROBLEMS

1. Find the center of gravity of the surface of a hemisphere.

Ans. $\frac{r}{2}$.

2. Given a spherical surface of radius r. A spherical segment of height h is cut off by a plane. Find the distance from this plane to the center of gravity of the surface of the spherical segment, including its plane base.

Ans. $\frac{rh}{4r-h}$.

3. Find the distance of the center of gravity of a lune from its axis, where 2α is the angle of the lune and r the radius of the sphere.

Ans. $\frac{\pi r}{4}\left(\frac{\sin\alpha}{\alpha}\right)$.

4. A thin silver bowl has the form of a spherical segment. Any radius of its circular top subtends an angle α at the center of the sphere. A semicircular lid of the same material and thickness covers half of the bowl. The bowl is placed on a smooth table. Find the angle ϕ between the plane of the lid and the table.

Ans. $\tan\phi = \frac{4\sin\alpha}{3\pi(2+\cos\alpha)}$.

45. Center of gravity of volumes. The center of gravity of the volume of a body coincides with the center of gravity of the body if the body is homogeneous. If the body is nonhomogeneous, the density of any element of volume must be expressed as a function of the coördinates of the elementary volume.

EXAMPLES

1. Find the center of gravity of a tetrahedron, or pyramid with triangular base.

Solution. The tetrahedron may be considered as made up of an infinite number of infinitely thin triangular plates parallel to the base. The center of gravity of each plate is at the intersection of the medians of the plate. Since the plates are similar and similarly placed, the center of gravity of each plate lies on a line joining the vertex of the tetrahedron with the center of gravity of its base.

The center of gravity of the tetrahedron lies on DH and on EC, and therefore at G, the point of intersection of DH and EC.

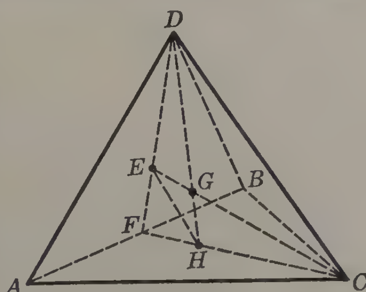

FIG. 94

From the similar triangles GEH and GCD,

$$\frac{GH}{GD}=\frac{EH}{DC}.$$

From the similar triangles EFH and DFC,

$$\frac{EH}{DC}=\frac{FE}{FD}=\frac{1}{3}.$$

Hence $GH = \frac{1}{3}\,GD$, from which $GH = \frac{1}{4}\,DH$.

The center of gravity of the tetrahedron therefore lies at a distance of one fourth the altitude from the base.

It follows that the equivalent particles for a tetrahedron are four particles, one at each vertex, each having a weight of one fourth the weight of the tetrahedron.

2. Find the center of gravity of any pyramid or cone.

Solution. A pyramid of any number of faces may be divided into tetrahedra by dividing its base into triangles. Since the center of gravity of each tetrahedron lies in a plane parallel to the base at a distance of one fourth of the common altitude from the base, the center of gravity of the pyramid lies in the same plane.

By reasoning similar to that in the preceding example, it follows that the center of gravity of any pyramid lies on the line joining its vertex to the center of gravity of the base.

Since any *cone*, right or oblique, may be divided into an infinite number of infinitesimal pyramids having a common vertex, it follows that the center of gravity of a cone lies one fourth of the way up the straight line joining the center of gravity of the base with the vertex.

3. Find the center of gravity of a portion of a sphere included between two parallel planes.

Solution. The elementary volume is $\pi y^2\,dx$, and its center of gravity is $(x, 0, 0)$.

Hence

$$\bar{x}=\frac{\int_k^h \pi(r^2-x^2)x\,dx}{\int_k^h \pi(r^2-x^2)\,dx}$$

$$=\frac{\frac{1}{2}r^2(h^2-k^2)-\frac{1}{4}(h^4-k^4)}{r^2(h-k)-\frac{1}{3}(h^3-k^3)}.$$

By making $k=0$ and $h=r$, the center of gravity of the hemisphere is $\bar{x}=\frac{3\,r}{8}$.

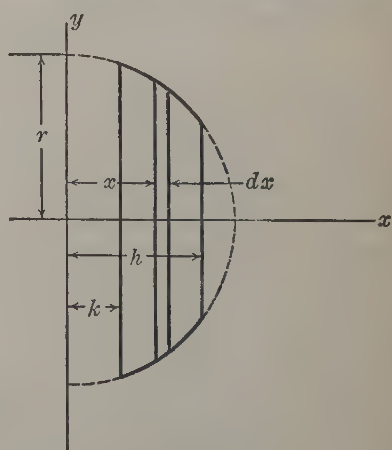

FIG. 95

4. Find the center of gravity of a segment of a right circular cone included between two concentric spheres whose centers coincide with the vertex of the cone.

Solution. Let the elementary volume dV be a spherical shell of radius ρ and thickness $d\rho$ and bounded by the conical surface.

Then $$dV = \int_{\theta=0}^{\theta=\alpha} 2\,\pi\rho \sin\theta \cdot \rho\, d\theta \cdot d\rho = 2\,\pi(1-\cos\alpha)\rho^2 \cdot d\rho.$$

The distance from the vertex of the cone to the center of gravity of this element of volume dV is, by § 44, Example 2,

$$\frac{\rho}{2}(1+\cos\alpha).$$

Hence, for the whole solid,

$$\bar{x} = \frac{\int_{r_1}^{r_2} 2\,\pi(1-\cos\alpha)\rho^2\, d\rho \cdot \frac{\rho}{2}(1+\cos\alpha)}{\int_{r_1}^{r_2} 2\,\pi(1-\cos\alpha)\,\rho^2\, d\rho}$$

$$= \frac{3}{8}\left(\frac{r_2^4 - r_1^4}{r_2^3 - r_1^3}\right)(1+\cos\alpha).$$

If $r_1 = 0$ and $r_2 = r$, the center of gravity of a cone bounded by a spherical base is given by

$$\bar{x} = \frac{3\,r}{8}(1+\cos\alpha).$$

When the semivertical angle of the cone becomes 90°, the solid becomes a hemisphere and

$$\bar{x} = \frac{3\,r}{8}.$$

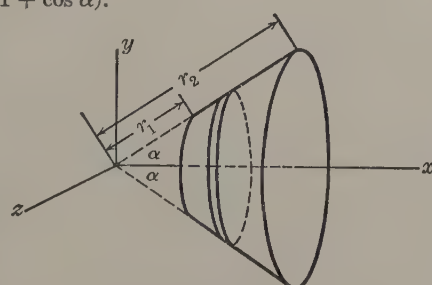

FIG. 96

PROBLEMS

1. The body shown in Fig. 97 consists of a hemisphere, a cylinder, and a cone. Find the distance from the vertex of the cone to the center of gravity of the body.

Ans. $2\frac{1}{3}$ in.

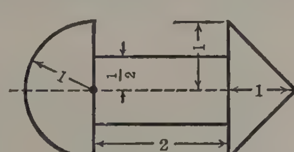

FIG. 97

2. Find the distance from the upper base to the center of gravity of a truncated right circular cone. The distance between the bases is h, and the radii of the upper and lower bases are r and R respectively.

Ans. $\frac{h}{4}\left(\frac{3\,R^2 + 2\,Rr + r^2}{R^2 + Rr + r^2}\right)$.

3. A solid cone whose semivertical angle is arc tan $\frac{1}{\sqrt{2}}$ is inscribed in a thin spherical shell. Show that there is no tendency for the shell to roll on a smooth, level table.

4. A sphere of radius r in. is removed from a sphere of radius 28 in., the distance between their centers being 14 in. The center of gravity of the remaining portion is 2 in. from the center of the larger sphere. Find the radius of the removed sphere. *Ans.* 14 in.

5. A hole of length h is drilled entirely through a hemisphere, the axis of the drill coinciding with the radius which is perpendicular to the base. Find the position of the center of gravity of the remaining portion. *Ans.* $\frac{3h}{8}$.

6. A pulley weighing 25 lb. has its center of gravity 0.024 in. from its geometric center. In order to make the center of gravity coincide with the geometric center a hole is drilled 6 in. from the center. How much metal should be removed? *Ans.* $\frac{1}{10}$ lb.

7. Find the distance from the upper base to the center of gravity of the prismoid shown in Fig. 98.

Ans. $\frac{h}{2}\left(\frac{3\,ab + a_1b_1 + a_1b + ab_1}{2\,ab + 2\,a_1b_1 + a_1b + ab_1}\right)$.

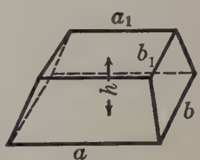

FIG. 98

8. Assuming the density of a line to vary as the first power or as the second power of the distance from one end of the line, find the distance of its center of gravity from the end. Interpret the results so as to obtain the center of gravity of (a) a triangular area, (b) the surface of a right cone, and (c) the volume of a right cone.

46. Theorems of Pappus. *If any plane area revolves through any angle α about an axis in its plane,*

(I) *The area of the surface generated by the perimeter of the plane area is equal to the product of the perimeter and the length of the arc described by the center of gravity of the perimeter.*

(II) *The volume of the solid generated by the plane area is equal to the product of the area and the length of the arc described by the center of gravity of the area.*

Proof of I. Let the x axis be the axis of revolution, and let the plane area lie initially in the xy plane. To avoid negative areas and volumes it is assumed that the plane area is not cut by the axis of revolution.

The area generated by an element of arc ds when the plane area rotates through an angle α is $\alpha y \cdot ds$; hence the total area generated is

$$A = \int \alpha y\, ds = \alpha \int y\, ds.$$

But
$$\bar{y} = \frac{\int y\,ds}{\int ds};$$

therefore
$$A = \alpha\bar{y}\int ds = \alpha\bar{y} \times \text{perimeter.}$$

Hence the area is equal to the product of the perimeter and the length of arc described by the center of gravity of the perimeter.

Proof of II. The volume generated by an element of area dA is $\alpha y\,dA$, where y is the ordinate of the element of area. The total volume generated is

$$V = \int \alpha y\,dA = \alpha \int y\,dA.$$

But
$$\bar{y} = \frac{\int y\,dA}{\int dA};$$

therefore
$$V = \alpha\bar{y}\int dA = \alpha\bar{y} \times \text{area.}$$

Hence the volume is equal to the product of the area and the length of the arc described by the center of gravity of the area.

When the center of gravity of an arc or an area is known, these theorems may be used to find the surface or volume of revolution generated by the arc or area, and conversely.

EXAMPLES

1. Find the surface and volume of a tore.

Solution. The solid may be generated by the *complete* revolution of a circle about an axis in its own plane.

Let r be the radius of the generating circle and R the distance of its center from the axis. The perimeter of the generating circle is $2\pi r$, and the length of arc described by the center of gravity of the perimeter is $2\pi R$. Hence the surface of the tore is $4\pi^2 rR$.

The area of the generating circle is πr^2, and the arc described by its center of gravity is $2\pi R$. Hence the volume of the tore is $2\pi^2 r^2 R$.

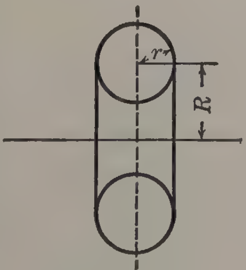

FIG. 99

2. A hemisphere may be regarded as generated by a semicircular area rotated 180° about its diameter. Find the center of gravity of the semicircular area.

Solution. The substitution of the proper values of V and A in the second theorem of Pappus,

$$V = \alpha\bar{y}A,$$

gives

$$\frac{2\,\pi r^3}{3} = \pi\bar{y}\left(\frac{\pi r^2}{2}\right),$$

from which

$$\bar{y} = \frac{4\,r}{3\,\pi}.$$

PROBLEMS

1. Find the center of gravity of the arc of a semicircle by means of the theorem of Pappus.

2. Find the volume of a solid sector of a sphere with a circular rim, and also the area of its curved surface. The vertical angle of the sector is $4\,\alpha$. *Ans.* $\frac{4}{3}\,\pi r^3 \sin^2\alpha$, $4\,\pi r^2 \sin^2\alpha$.

3. A solid is generated by the revolution of a triangle ABC, of altitude h, about the side AB. Find the surface and volume generated. *Ans.* $\pi(a+b)h$, $\frac{1}{3}\,\pi ch^2$.

CHAPTER VI

PLANE STATICS OF A RIGID BODY

47. Resultant of nonconcurrent coplanar forces. Let a system of coplanar forces P_1, P_2, $\cdots$ act upon a rigid body at points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $\cdots$, and let their lines of action make angles α_1, α_2, $\cdots$ with the positive direction of a system of rectangular axes.

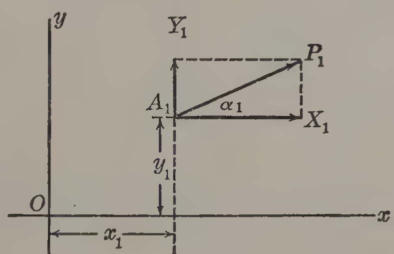

FIG. 100

The forces P_1, P_2, $\cdots$ may be replaced by their rectangular components X_1, Y_1, X_2, Y_2, $\cdots$.

By § 36 the force X_1 acting at A_1 may be replaced by an equal parallel force acting at O together with a couple whose moment is $-X_1y_1$.

Similarly, the force Y_1 may be replaced by an equal parallel force acting at O together with a couple whose moment is $+Y_1x_1$.

The components X_2, Y_2, $\cdots$ may similarly be replaced by parallel forces at the origin together with the proper couples.

Thus the original system of forces P_1, P_2, $\cdots$ may be replaced by

(1) a force $\Sigma X = X_1 + X_2 + \cdots$ along the x axis, acting at O,

(2) a force $\Sigma Y = Y_1 + Y_2 + \cdots$ along the y axis, acting at O,

and

(3) a couple $\Sigma M = \Sigma(Yx - Xy) = (Y_1x_1 - X_1y_1) + (Y_2x_2 - X_2y_2) + \cdots$ in the xy plane.

By § 20 the forces at O may be replaced by a resultant R acting at O and making an angle θ with the positive x axis, where

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$$

and

$$\tan\theta = \frac{\Sigma Y}{\Sigma X}.$$

Hence, finally, *any system of coplanar forces may be reduced to or replaced by a single force R, acting at an arbitrary origin, and a couple ΣM lying in the plane.*

Furthermore, since a force and a couple in the same plane may, by § 35, be reduced to a *single force*, it follows that *in general* the force R and the couple ΣM may be replaced by a single force R equal and parallel to R at a distance p from its former position through O, where

$$p = \frac{\Sigma M}{R}.$$

The direction in which p is measured from R is determined from the algebraic sign of the couple ΣM.

In particular cases it may happen either that R or ΣM is zero or that R and ΣM both vanish. It is evident that if only R vanishes, the resultant of the system of forces is a couple whose moment is an invariable quantity, independent of the choice of origin. If only $\Sigma M = 0$, the system reduces to a single resultant force R which passes through the origin. If $R = 0$ and $\Sigma M = 0$, the given system of forces is equivalent to no force; in other words, the forces exactly balance each other, and the system is in equilibrium.

48. The base point. The arbitrary point O of § 47 to which the forces of the system were transferred is frequently called the *base point.*

The method of obtaining the resultant R of § 47 does not involve the use of the coördinates of the points of application of the forces. The resultant R is therefore invariable in magnitude and direction, whatever base point is chosen. Furthermore, the single force R, obtained by combining the resultant R and the couple ΣM, is an invariable quantity independent of the choice of the base point or the direction of axes.

On the contrary, the expression for the resultant couple, ΣM, involves the coördinates of the points of application of the forces, and consequently the magnitude of the couple ΣM is, *in general,* different for different base points. When, however, the system reduces to a couple (that is, when $R = 0$), the couple is an invariable quantity, independent of the choice of the base point or the direction of axes.

49. Conditions of equilibrium. A system of coplanar forces acting upon a rigid body is in equilibrium when the forces

considered collectively have no tendency to translate or rotate the body. Experience teaches that these conditions are fulfilled when and only when the resultant force R and the resultant couple ΣM both vanish.

The condition that $R = 0$ necessitates both that $\Sigma X = 0$ and that $\Sigma Y = 0$, since $(\Sigma X)^2$ and $(\Sigma Y)^2$ are positive quantities.

The necessary and sufficient conditions under which a system of coplanar forces acting on a rigid body is in equilibrium are

$$\Sigma X = 0,$$
$$\Sigma Y = 0,$$
$$\Sigma M = 0.$$

Otherwise expressed,

I. *The algebraic sums of the components of the forces along any arbitrary x and y rectangular axes must separately be zero.*

II. *The algebraic sum of the moments of the forces about an arbitrary point in their plane must be zero.*

It is evident that an infinite number of equations of equilibrium may be written by selecting different directions for the axes and different base points. It can be shown, however, that only three of these equations are independent.

50. Alternative conditions of equilibrium. The two following sets of conditions of equilibrium for any system of coplanar forces are advantageous in some cases:

I. *Any system of coplanar forces is in equilibrium if the sum of the moments is zero for each of two base points and if the sum of the components is zero in any one direction not perpendicular to the line joining the two base points.*

By § 47 any system of coplanar forces may be reduced to either (1) a single couple and no force or (2) a single force and no couple.

CASE 1. When there is no resultant force the couple is invariable, by § 48. Since by hypothesis the couple is zero for one base point, it is zero for all base points. Hence the system is in equilibrium.

CASE 2. Since there is no couple, the single force must pass through both base points in order that the sum of the moments about each base point shall be zero. If the component of this force in any direction not perpendicular to itself is zero, the force must vanish, and the system is in equilibrium.

II. *Any system of coplanar forces is in equilibrium if the sum of the moments is zero for each of three base points not in the same straight line.*

It is evident that there can be no resultant couple. There can be no resultant force, since its line of action must contain three base points not in a line. Therefore the system is in equilibrium.

51. Equilibrium of two forces acting on a body. If a rigid body is in equilibrium under the action of two forces, it is evident that one force is the anti-resultant, or equilibrant, of the other. The two forces must be equal in magnitude, be opposite in direction, and lie in the same straight line. If they were parallel and not in the same straight line, they would form a couple.

52. Equilibrium of three coplanar forces acting upon a rigid body. If a rigid body is in equilibrium under the action of three coplanar forces, their lines of action must meet in a point or be parallel.

Let P, Q, and R be three coplanar forces which are in equilibrium, and let the forces P and R intersect in a point O. Since the forces P, Q, and R are in equilibrium, the sum of their moments about O must be zero. Therefore the force Q must pass through O.

If the two forces P and R are parallel and unequal, the force Q must be equal and opposite to the resultant of the forces P and R.

The forces P and R cannot form a couple, because this couple cannot be in equilibrium with the single force Q.

53. The solution of problems. The steps which lead to the solution of a problem in plane statics may be formulated as follows:

1. Make a sketch showing
   (*a*) the body or bodies and the geometric configuration;
   (*b*) their contacts, attachments, weights, etc.

2. Select *one of the bodies*, or consider any group of the bodies as *one body*. Make a separate sketch of the selected body and choose a set of rectangular axes. Represent the known forces (including the weight) acting upon the selected body by arrows or vectors. Represent each unknown constraint (pin reac-

tion, contact reaction, attachment, etc.) by its two rectangular components. These rectangular component forces may be drawn in either the positive or the negative direction, but when once selected they must be adhered to throughout the problem.

This separate sketch of the selected body, including all the forces acting upon it, is called the *free-body* diagram.

3. Impose the conditions of equilibrium ($\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma M = 0$) upon the forces acting on the free body.

4. Select another body and proceed as in steps 2 and 3, noting carefully that the force which a body A exerts on a body B is numerically equal but opposite in direction to the force which the body B exerts upon the body A. Continue this process until a sufficient number of equations are obtained to determine the unknown forces.

EXAMPLES

1. A homogeneous rod of weight W and length $2\,l$ is suspended by a string attached to its upper end, as shown in Fig. 101. The lower end rests on a smooth horizontal table. Find the pressure on the table, the tension in the string, and the angle which the string makes with the vertical.

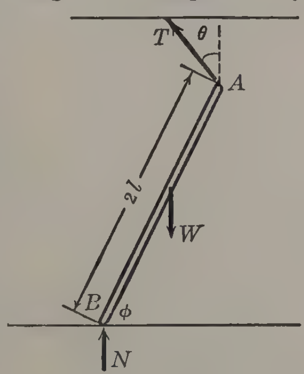

FIG. 101

Solution. The rod AB is the free body. The forces acting upon it are T, N, and W, as shown in Fig. 101. There is no horizontal component at B, since it is assumed that there is no friction between the end of the rod and the table. The three conditions of equilibrium are

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \text{and} \quad \Sigma M = 0.$$

$$\Sigma X = 0 \text{ gives } -T \sin\theta = 0. \tag{1}$$

$$\Sigma Y = 0 \text{ gives } N + T\cos\theta - W = 0. \tag{2}$$

$$\Sigma M_A = 0 \text{ gives } Wl\cos\phi - 2\,Nl\cos\phi = 0. \tag{3}$$

From (1), $\theta = 0.$

From (3), $N = \dfrac{W}{2}.$

From (2), $T = \dfrac{W}{2}.$

2. A uniform rod of length l and weight W, situated in a vertical plane, is inclined at an angle α to the horizontal. The upper end B rests against a smooth wall inclined at an angle β to the horizontal. The lower end is supported on a smooth horizontal plane, and the rod is maintained in equilibrium by a horizontal force A_H applied at the lower end of the rod, as shown in Fig. 102. Find the reactions at the upper and lower ends of the rod.

Solution. Since the wall is smooth the reaction Q is normal to the wall. Imposing the conditions of equilibrium,

$$\Sigma X = 0 \text{ gives } A_H - Q \sin \beta = 0. \tag{1}$$

$$\Sigma Y = 0 \text{ gives } A_V + Q \cos \beta - W = 0. \tag{2}$$

$$\Sigma M_A = 0 \text{ gives } Ql \cos(\beta - \alpha) - W \frac{l}{2} \cos \alpha = 0. \tag{3}$$

From (3), $$Q = \frac{W \cos \alpha}{2 \cos(\beta - \alpha)}.$$

From (2), $$A_V = W\left[1 - \frac{\cos \alpha \cos \beta}{2 \cos(\beta - \alpha)}\right].$$

From (1), $$A_H = \frac{W \cos \alpha \sin \beta}{2 \cos(\beta - \alpha)}.$$

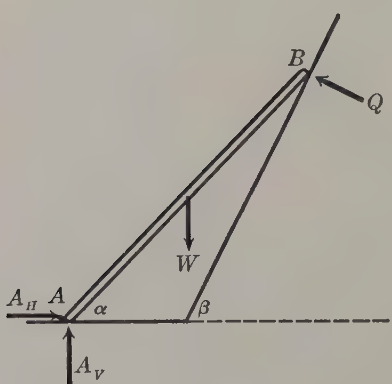

FIG. 102

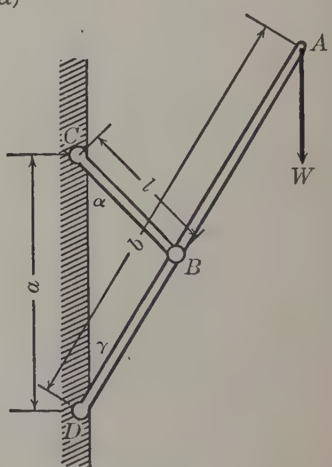

FIG. 103

3. Two rods, AD and BC, whose weights may be neglected, are attached by smooth pins to a vertical wall at C and D, and are joined by a smooth pin at B, as shown in Fig. 103. A weight W is attached to the rod AD at A. If the distance $AD = b$, $DC = a$, and the angles which the rods make with the wall are α and γ respectively, find the magnitude and direction of the pin reactions at B and D.

Solution. For the free body AD, shown in Fig. 104,

$$\Sigma M_D = 0 \text{ gives } Wb \sin\gamma - B_H x \cos\gamma - B_V x \sin\gamma = 0. \qquad (1)$$

$$\Sigma X = 0 \text{ gives } D_H - B_H = 0. \qquad (2)$$

$$\Sigma Y = 0 \text{ gives } B_V - W - D_V = 0. \qquad (3)$$

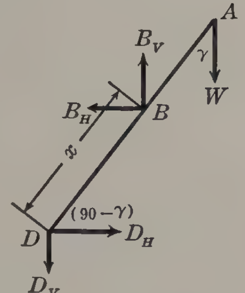

FIG. 104

For the free body BC, shown in Fig. 105, $\Sigma M_C = 0$ gives

$$B_H l \cos\alpha - B_V l \sin\alpha = 0. \qquad (4)$$

From the triangle BCD, by the Sine Law,

$$\frac{x}{\sin\alpha} = \frac{a}{\sin(\alpha+\gamma)}. \qquad (5)$$

Substituting the value of B_H from (4) and the value of x from (5) in (1) gives

$$Wb\sin\gamma - B_V \tan\alpha\left[\frac{a\sin\alpha}{\sin(\alpha+\gamma)}\right]\cos\gamma - B_V\left[\frac{a\sin\alpha}{\sin(\alpha+\gamma)}\right]\sin\gamma = 0,$$

from which $\qquad B_V = \dfrac{Wb}{a}\sin\gamma\cot\alpha.$

From (4), $\qquad B_H = \dfrac{Wb}{a}\sin\gamma.$

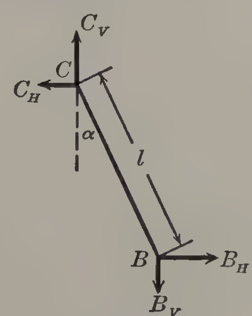

FIG. 105

The magnitude of the pin reaction at B is

$$\sqrt{B_V^2 + B_H^2} = \frac{Wb}{a}\sin\gamma\csc\alpha,$$

and the angle which this reaction makes with the vertical is given by

$$\tan\theta_B = \frac{B_H}{B_V} = \tan\alpha.$$

Therefore $\theta_B = \alpha$, and hence the force acting on the rod BC acts along the axis of the rod.

Furthermore the pin reaction at C is equal and opposite to the pin reaction at B.

From (2), $\qquad D_H = B_H = \dfrac{Wb}{a}\sin\gamma.$

From (3), $\qquad D_V = B_V - W = W\left(\dfrac{b}{a}\sin\gamma\cot\alpha - 1\right).$

The magnitude of the pin reaction at D is

$$\sqrt{D_H^2 + D_V^2} = W\sqrt{\frac{b^2}{a^2}\sin^2\gamma\csc^2\alpha - 2\frac{b}{a}\sin\gamma\cot\alpha + 1},$$

and the angle which this reaction makes with the vertical is given by

$$\tan\theta_D = \frac{D_H}{D_V} = \frac{b\sin\gamma}{b\sin\gamma\cot\alpha - a}.$$

4. The crane shown in Fig. 106 is made of uniform beams weighing 40 lb. per foot joined together by smooth pins at C, E, and F. The total weight of the crane and its load of 2000 lb. is supported at A. Find the reactions at A and B and the pin reactions at C, E, and F, in magnitude and direction.

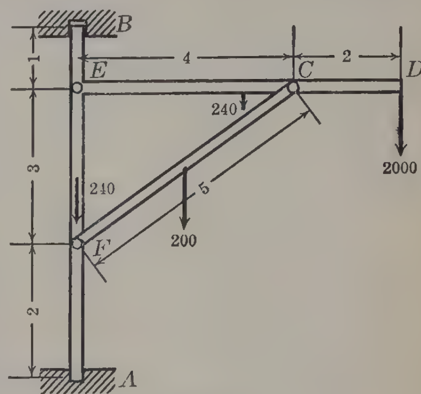

FIG. 106

Solution. Select the whole crane as the free body. The known forces acting upon the free body are the 2000-pound weight and the weights of the three beams. The unknown forces are the reactions shown at A and B. The force which the beam FC exerts on the pin at C is equal and opposite to the force which the beam ED exerts on the pin at C. Therefore the two forces acting on the pin at C or at E or F cancel each other and do not enter into the equations of equilibrium when the whole crane is selected as the free body.

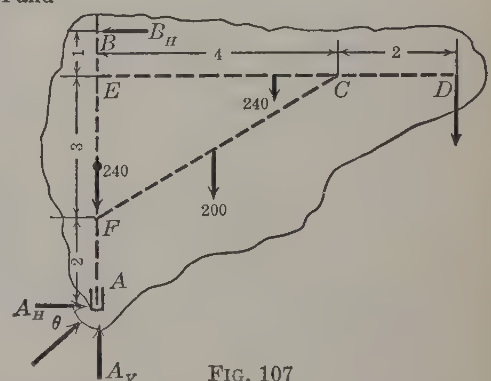

FIG. 107

Imposing the conditions of equilibrium,

$$\Sigma X = 0 \text{ gives } A_H - B_H = 0. \tag{1}$$

$$\Sigma Y = 0 \text{ gives } A_V - 2000 - 240 - 200 - 240 = 0. \tag{2}$$

$$\Sigma M_A = 0 \text{ gives } 6\,B_H - (2000)6 - (240)3 - (200)2 = 0. \tag{3}$$

From (3), $B_H = 2187$ lb.

From (1), $A_H = 2187$ lb.

From (2), $A_V = 2680$ lb.

The resultant force A_R acting at A is equal to $\sqrt{A_H^2 + A_V^2} = 3459$ lb. The angle θ which the resultant A_R makes with the positive x axis is given by

$$\tan\theta = \frac{A_V}{A_H} = \frac{2680}{2187} = 1.225,$$

from which $\theta = 50°\,47'$.

In order to determine the pin reactions at C, E, and F, the free-body diagrams of the beam ED and also the beam CF are drawn.

Imposing the conditions of equilibrium upon the forces acting on the free body *ED* Fig. 108,

$$\Sigma X = 0 \text{ gives } C_H - E_H = 0. \tag{4}$$
$$\Sigma Y = 0 \text{ gives } C_V + E_V - 2000 - 240 = 0. \tag{5}$$
$$\Sigma M_E = 0 \text{ gives } 4\,C_V - (240)3 - (2000)6 = 0. \tag{6}$$

Imposing the conditions of equilibrium upon the forces acting on the free body *FC*, Fig. 109,

$$\Sigma X = 0 \text{ gives } F_H - C_H = 0. \tag{7}$$
$$\Sigma Y = 0 \text{ gives } F_V - C_V - 200 = 0. \tag{8}$$
$$\Sigma M_F = 0 \text{ gives } 3\,C_H - 4\,C_V - (200)2 = 0. \tag{9}$$

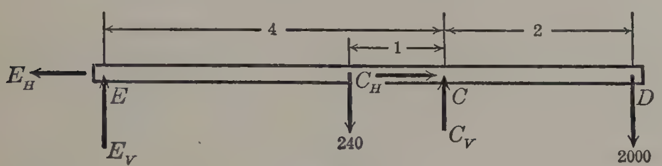

FIG. 108

From (6), $C_V = 3180$ lb.
Substituting the value of C_V in (5) gives

$$E_V = -940 \text{ lb.}$$

The negative value of E_V shows that the direction of the force was incorrectly assumed in Fig. 108. The external force E_V which the pin exerts on the beam really acts downward.

Substituting the value of C_V in (8) gives

$$F_V = 3380 \text{ lb.}$$

Substituting the value of C_V in (9) gives

$$C_H = 4373 \text{ lb.}$$

From (7),

$$F_H = 4373 \text{ lb.}$$

From (4),

$$E_H = 4373 \text{ lb.}$$

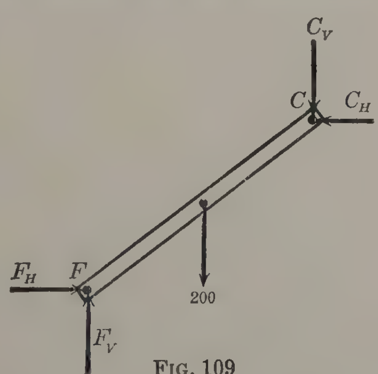

FIG. 109

The values of the components of the pin reactions may be collected and the magnitude and direction of the resultant reactions obtained. Thus,

$$\left.\begin{aligned} E_H &= 4373 \text{ lb.} \\ E_V &= -940 \text{ lb.} \end{aligned}\right\} E_R = 4473 \text{ lb.,} \quad \tan\theta_E = .2150, \quad \theta_E = 12°\,8'.$$

$$\left.\begin{aligned} C_H &= 4373 \text{ lb.} \\ C_V &= 3180 \text{ lb.} \end{aligned}\right\} C_R = 5407 \text{ lb.,} \quad \tan\theta_C = .7272, \quad \theta_C = 36°\,2'.$$

$$\left.\begin{aligned} F_H &= 4373 \text{ lb.} \\ F_V &= 3380 \text{ lb.} \end{aligned}\right\} F_R = 5527 \text{ lb.,} \quad \tan\theta_F = .7729, \quad \theta_F = 37°\,42'.$$

PROBLEMS

1. A uniform rod 4 ft. long, weighing 10 lb., is hinged at its lower end and makes an angle of 30° with the horizontal. The upper end of the rod rests against a smooth vertical wall. Find the magnitude and direction of the hinge reaction and the pressure on the wall.

Ans. 13.23 lb., 49° 6′, 8.66 lb.

2. A uniform bar of length $2\,l$ and weight W, making an angle α with the horizontal and lying in a vertical plane, rests in a box of width a, as shown in Fig. 110. Find the vertical and horizontal components of the reaction at A and the reaction at B.

Ans. $W\left(1 - \frac{l}{a}\cos^3\alpha\right)$,

$W\frac{l}{a}\cos^2\alpha\sin\alpha$,

$W\frac{l}{a}\cos^2\alpha$.

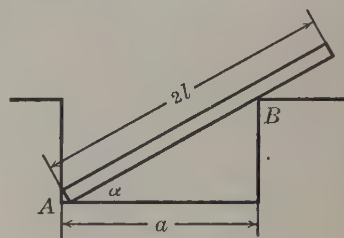

Fig. 110

3. Find the reactions at A and B and the pin reactions at C, E, and F of the crane shown in Fig. 106, if the members weigh 50 lb. per foot and if the load at D is replaced by a load of 1000 lb. and an additional load of 500 lb. is suspended from the middle point of CF.

Ans. $A_V = 2350$ lb., $E_R = 2833$ lb.,
$B_H = A_H = 1400$ lb., $C_R = 3289$ lb.,
$\theta = \text{arc tan } \frac{47}{28}$, $F_R = 3737$ lb.

4. A cylinder of weight W and radius r is held on an inclined plane by a parbuckle. If the inclined plane makes an angle θ with the horizontal and the upper portion of the parbuckle is horizontal, find the tension.

Ans. $T = W \tan\frac{\theta}{2}$.

5. Find the minimum tension T in the rope necessary to start cylinder B over the block A as shown in Fig. 111.

Ans. 50 lb.

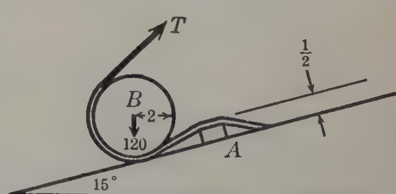

Fig. 111

6. A light rod 8 ft. long, carrying particles at its extremities weighing 5 lb. and 10 lb. respectively, is placed inside a smooth, fixed, hollow sphere 10 ft. in diameter. Find the inclination of the rod to the horizontal in its equilibrium position.

Ans. arc tan $\frac{4}{9}$.

7. A uniform bar AD 12 ft. long, weighing 72 lb. and inclined 30° to the horizontal, is held in position in a vertical plane against a smooth horizontal floor by two smooth pegs, B and C, 4 ft. apart, as shown in Fig. 112. Find the pressures at A, B, and C. *Ans.* 72 lb., 93.53 lb., 93.53 lb.

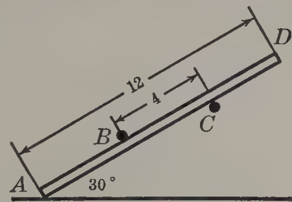

FIG. 112

8. A bar AB weighing 100 lb., 8 ft. long, whose center of gravity is 2 ft. from the lower end, rests upon a smooth wall and floor as shown in Fig. 113. Motion is prevented by a cord AC. Find the pressure on the floor at A, the tension in the cord, and the normal pressure at B. *Ans.* 87.50 lb., 21.65 lb., 25 lb.

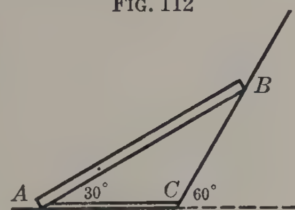

FIG. 113

9. A derrick boom AB of weight W and length l is hinged at the lower end B and held in position by a cable which passes over a pulley fixed at C, where $BC = AB$, as shown in Fig. 114. Find the tension T in the cable when the boom is in equilibrium and makes an angle α with the vertical. Find the pin reaction at B in magnitude and direction.

$$\textit{Ans. } T = W \sin\frac{\alpha}{2},\ R_B = W\cos\frac{\alpha}{2},\ \theta = \frac{\alpha}{2}.$$

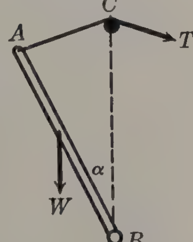

FIG. 114

10. Two uniform bars, AB of length 20 ft. and weight 144 lb., and BC of length 2.5 ft. and weight 25 lb., rest in a smooth rectangular box as shown in Fig. 115. If the angle ABC is 90°, what must be the width of the box in order that the bars may be in equilibrium? *Ans.* 10 ft.

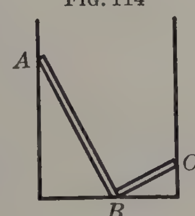

FIG. 115

11. Two bars, AB and AC, are hinged together at A, supported by hinge joints at fixed points B and C, and loaded as shown in Fig. 116. Find the horizontal and vertical components of the hinge reactions at A, B, and C.

Ans. $A_H = 132$ lb.,
$A_V = 126$ lb.,
$B_H = 132$ lb.,
$B_V = 226$ lb.,
$C_H = 132$ lb.,
$C_V = 274$ lb.

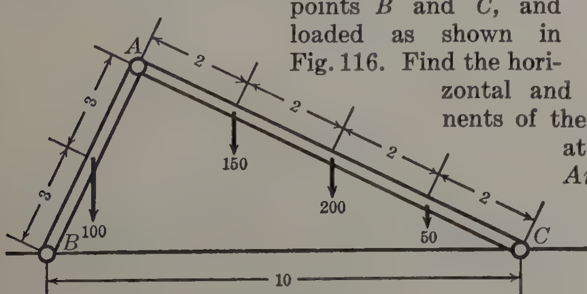

FIG. 116

12. A smooth sphere of weight 10 lb. and radius 2 ft. is suspended from a fixed point O by a string 1 ft. long attached to its surface. A 4-pound weight is attached to O by a string passing over the sphere. Find the angle between the string which holds the sphere and the vertical. *Ans.* $\theta = \text{arc sin } \frac{4}{21}$.

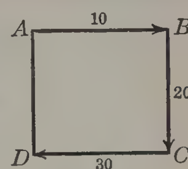

FIG. 117

13. The forces shown in Fig. 117 act along the edges of a block 2 ft. square. Find the resultant force in magnitude and direction, and find the distance from C to its line of action. *Ans.* 28.28 lb., 225°, 0.71 ft.

14. Three uniform bars of equal length are jointed at the ends to form an equilateral triangle ABC, which is suspended from the vertex A. The bars AB and AC weigh 10 lb. each, and the bar BC weighs 20 lb. Find the reaction at B in magnitude and direction. *Ans.* $5\sqrt{7}$ lb., $\tan \theta = \frac{2}{3}\sqrt{3}$.


FIG. 118

15. Two equal uniform rods, of length 6 ft. and weight 20 lb., are pinned together at their middle points and support a cylinder of radius 6 in. and weight 100 lb., the upper ends of the rods being joined by a string, as shown in Fig. 118. If the vertical angle between the rods is 29°, find the tension in the string. *Ans.* $T = 151.0$ lb.

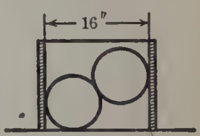

FIG. 119

16. Two equal spheres, of weight 10 lb. and radius 6 in., are placed in a thin cylindrical tube of radius 8 in., as shown in Fig. 119. What is the least weight of the tube in order that it may not tip over? *Ans.* 5 lb.

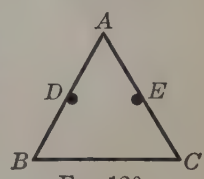

FIG. 120

17. Three rods, each weighing 20 lb. and 4 ft. in length, are pin-jointed to form an equilateral triangle. The triangle rests in equilibrium on two smooth pegs 2 ft. apart, as shown in Fig. 120. Show that the reaction at A is horizontal and equal to 28.9 lb., the reaction at D is 60 lb., and the reaction at B is 25.2 lb.

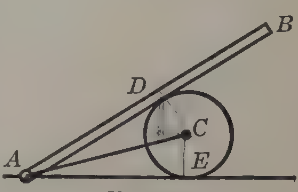

FIG. 121

18. A uniform rod AB of length 26 in. and weight 169 lb. is hinged to a smooth floor at A and rests upon a smooth cylinder of radius 5 in. and weight 20 lb. The cylinder is kept in position by a string AC of length 13 in., as shown in Fig. 121. Find the tension in the string. *Ans.* $99\frac{1}{6}$ lb.

19. A uniform rod of weight W and length $2a$ rests in a smooth hemispherical bowl of radius r, as shown in Fig. 122. Find the angle θ when the rod is in equilibrium and also the reactions at the points of contact of rod and bowl.

$$Ans.\ \cos\theta = \frac{a \pm \sqrt{a^2 + 32r^2}}{8r},$$

$$A = W \tan\theta,\ C = \frac{Wa}{2r}.$$

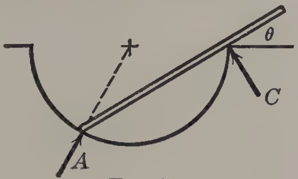

FIG. 122

20. Four equal rods, each of weight 10 lb. and length 2 ft., are pin-jointed to form a rhombus, as shown in Fig. 123. The rhombus is suspended from the vertex A, and the vertices C and A are connected by a string, making the angle BAD equal to 60°. Find the tension in the string and the reaction at B.

HINT. Consider the pin at A as a free body. Take moments about B for the rods AB and BC.

Ans. 20 lb., 2.89 lb. horizontal.

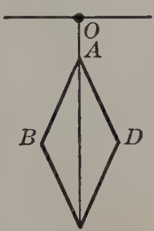

FIG. 123

21. A uniform bar AB of length $2l$ and weight W is placed in a thin, hollow, smooth hemisphere of radius r and weight S, which rests on a horizontal plane, as shown in Fig. 124. In the position of equilibrium the bar contacts the inner surface of the sphere at A and rests on the edge at C in a horizontal position. Show that $S : W = 2l : \sqrt{4r^2 - l^2}$. What are the reactions at A, C, and D?

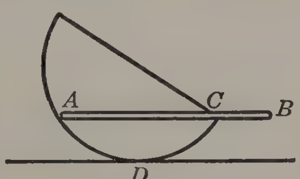

FIG. 124

22. A rectangular board $ABCD$, 12 in. × 15 in., is suspended by a string 12 in. long attached to the vertex A. The string is attached to a smooth vertical wall so that the plane of the board is perpendicular to the wall and the vertex D rests against the wall, as shown in Fig. 125. Find the distances from A, B, and C to the wall in the position of equilibrium.

Ans. 4.61 in., 18.46 in., and 13.84 in.

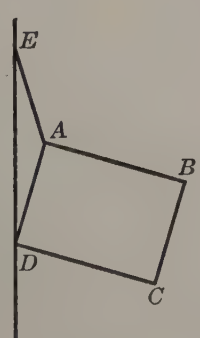

FIG. 125

23. A solid hemisphere weighing 24 lb. rests with its convex surface on a level table. A particle weighing 3 lb. is attached to a point in the circumference of the plane face of the hemisphere. Find the angle between the table and the face of the hemisphere in its equilibrium position.

Ans. $\tan\theta = \frac{1}{3}$.

24. Two smooth cylinders, of weights W_1 and W_2, rest in contact in the angle between two planes inclined at angles α and β to the horizontal, as shown in Fig. 126. Find the reactions upon the planes, the pressure between the cylinders, and the angle θ which the line of centers of the cylinders makes with the vertical when equilibrium exists.

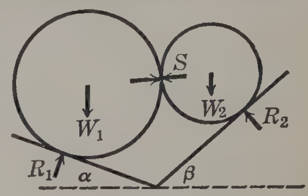

FIG. 126

Ans. $R_1 = \dfrac{(W_1 + W_2)\sin\beta}{\sin(\alpha+\beta)}$, $R_2 = \dfrac{(W_1 + W_2)\sin\alpha}{\sin(\alpha+\beta)}$,

$$S = \sqrt{R_1^2 - 2\,R_1W_1\cos\alpha + W_1^2},\ \cot\theta = \frac{W_2\cot\alpha - W_1\cot\beta}{W_1 + W_2}.$$

25. A cord 6 ft. long is attached to two points, one of which is 4 ft. to the right and 2 ft. above the other. A heavy ring which may slide on the cord without friction rests in its equilibrium position. What are the lengths of the segments into which the ring divides the cord? *Ans.* 1.66 ft., 4.34 ft.

CHAPTER VII

FORCES IN THREE DIMENSIONS

54. Fundamental theorems. Any system of forces *in a plane* may be reduced to a force and a couple in the same plane or to a single force (§ 47).

Any system of forces *in space* may be reduced to a force and a couple or to two forces. The force and couple do not in general lie in the same plane. The two forces in general cross each other in space without intersecting.

55. Resultant of a system of forces in space. Let P_1, P_2, $\cdots$ be any number of forces acting upon a rigid body at the points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$, $\cdots$ and let their direction angles be $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $\cdots$ respectively.

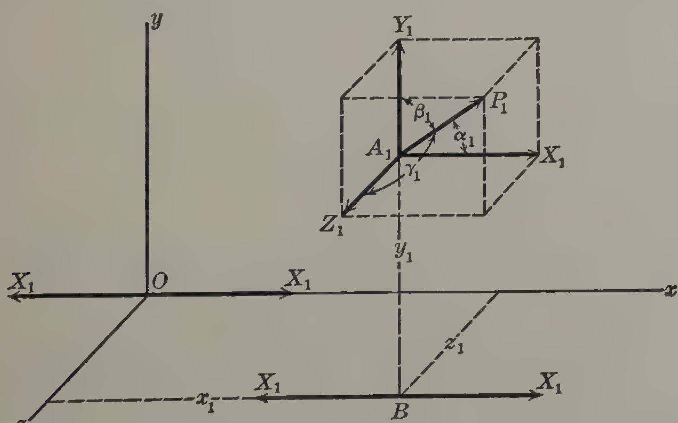

FIG. 127

Let O be an arbitrary point which is taken as the origin of a rectangular system of coördinates. Consider a single force P_1 at $A_1(x_1, y_1, z_1)$, which is typical of the remaining forces of the system. The force P_1 may be replaced by its rectangular components X_1, Y_1, Z_1, where $X_1 = P_1 \cos \alpha_1$, $Y_1 = P_1 \cos \beta_1$, and $Z_1 = P_1 \cos \gamma_1$.

The introduction of two equal and opposite forces at B and also at O, each numerically equal and parallel to X_1 at A_1, has no effect upon the system of forces. The forces X_1 at A_1 and $-X_1$ at B form a couple whose axis is parallel to the z axis and whose moment is $-X_1y_1$. The forces X_1 at B and $-X_1$ at O form a couple whose axis is parallel to the y axis and whose moment is X_1z_1.

The force X_1 at A_1 is thus replaced by

(1) an equal parallel force X_1 at O,
(2) a couple $-X_1y_1$ whose axis is parallel to the z axis, and
(3) a couple X_1z_1 whose axis is parallel to the y axis.

Similarly the force Y_1 at A_1 is replaced by

(1) a force Y_1 at O,
(2) a couple $-Y_1z_1$ whose axis is parallel to the x axis, and
(3) a couple Y_1x_1 whose axis is parallel to the z axis.

Similarly the force Z_1 at A_1 is replaced by

(1) a force Z_1 at O,
(2) a couple $-Z_1x_1$ whose axis is parallel to the y axis,
(3) a couple Z_1y_1 whose axis is parallel to the x axis.

The force P_1 at A_1 is therefore replaced by

(1) its components, X_1, Y_1, Z_1, at the point O,
(2) a couple whose moment is $Z_1y_1 - Y_1z_1$ and whose axis is parallel to the x axis,
(3) a couple whose moment is $X_1z_1 - Z_1x_1$ and whose axis is parallel to the y axis,
(4) a couple whose moment is $Y_1x_1 - X_1y_1$ and whose axis is parallel to the z axis.

When this process is repeated for each force, $P_2, P_3, \cdots$, the original forces are reduced to forces ΣX, ΣY, and ΣZ along the axes and couples L, M, and N about the x, y, and z axes respectively, where

$$\Sigma X = X_1 + X_2 + \cdots,$$
$$\Sigma Y = Y_1 + Y_2 + \cdots,$$
$$\Sigma Z = Z_1 + Z_2 + \cdots,$$
$$L = \Sigma(Zy - Yz) = (Z_1y_1 - Y_1z_1) + (Z_2y_2 - Y_2z_2) + \cdots,$$
$$M = \Sigma(Xz - Zx) = (X_1z_1 - Z_1x_1) + (X_2z_2 - Z_2x_2) + \cdots,$$
$$N = \Sigma(Yx - Xy) = (Y_1x_1 - X_1y_1) + (Y_2x_2 - X_2y_2) + \cdots.$$

The component forces ΣX, ΣY, and ΣZ may be combined into a single resultant R at O, where

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.$$

The direction cosines l, m, and n of the resultant R are given by the equations

$$l = \frac{\Sigma X}{R}, \quad m = \frac{\Sigma Y}{R}, \quad \text{and} \quad n = \frac{\Sigma Z}{R},$$

which may also be written

$$l = \frac{X}{R}, \quad m = \frac{Y}{R}, \quad \text{and} \quad n = \frac{Z}{R}.$$

The couples L, M, and N may be represented by vectors along the x, y, and z axes respectively, and they may be combined into a single resultant couple of moment G, where

$$G = \sqrt{L^2 + M^2 + N^2}.$$

The direction cosines of the vector which represents the couple G are

$$l' = \frac{L}{G}, \quad m' = \frac{M}{G}, \quad \text{and} \quad n' = \frac{N}{G}.$$

The plane of the couple G is at right angles to the vector whose direction cosines are l', m', and n'.

The force R and the couple G are called the *principal force* and the *principal couple* at the point O.

The method of obtaining the principal force R does not involve the use of the coördinates, and therefore R is invariable in magnitude and direction whatever point O is chosen for the base point.

On the contrary, the expression for G involves the coördinates, and hence the magnitude and direction of the principal couple G are in general different for different base points. However, if the *direction* and *magnitude* of the couple are maintained, it may be moved about in space without altering its effect. By selecting a proper position for the couple, one of the forces of the couple may be combined with the resultant R, thereby reducing the system to two forces whose lines of action cross in space without intersecting.

56. A wrench. Another important reduction of the principal force R and the principal couple G may be made. Let the vector G be moved parallel to itself so that it is laid off from the origin, or the base point O. The vectors G and R then

determine a plane. The vector G is resolved into its components parallel and perpendicular to R. The component parallel to R is $G \cos \theta$, and it represents a couple whose plane is perpendicular to R. The direction cosines of this plane are the same as the direction cosines of R. The couple $G \cos \theta$ tends to produce rotation about R as an axis.

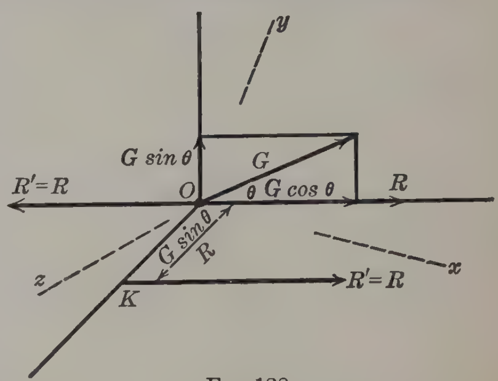

FIG. 128

The component of G perpendicular to R is $G \sin \theta$, and its plane may be shifted parallel to itself so as to contain R. The couple $G \sin \theta$ is reduced to a couple whose forces are each R and whose arm is $\frac{G \sin \theta}{R}$. This new form of the couple is shifted in its plane so that one of its forces R annuls the principal force R, leaving a single force R. The new force R, marked R' in Fig. 128, is equal to and parallel to the original R and lies in a plane which is perpendicular to the plane of R and G. The distance between R and R' is $\frac{G \sin \theta}{R}$.

The principal force R and the principal couple G are therefore reduced to a force $R' = R$, together with a couple $G \cos \theta$, whose axis coincides with R'.

The system composed of a force and a couple about that force as an axis is called a *wrench.*

57. Geometry of the wrench. From § 55, the direction cosines of R and G are l, m, n, and l', m', n' respectively. Therefore

$$\cos \theta = ll' + mm' + nn'.$$

The direction cosines a, b, and c of the plane containing R and G are given by the equations

$$\frac{a}{mn' - m'n} = \frac{b}{l'n - ln'} = \frac{c}{lm' - l'm}$$

and

$$a^2 + b^2 + c^2 = 1.$$

Hence the equation of the plane of R and G is $ax + by + cz = 0$.

The line OK is normal to the plane of R and G, and therefore the direction cosines of the line OK are a, b, c. Let (x', y', z') be the coördinates of the point K; then, since the length of OK is $\frac{G \sin \theta}{R}$, it follows that

$$x' = \frac{aG \sin \theta}{R}, \quad y' = \frac{bG \sin \theta}{R}, \quad \text{and} \quad z' = \frac{cG \sin \theta}{R}.$$

Since the force R' of the wrench is parallel to the principal force R, its direction cosines are l, m, n respectively. The equation of the axis or line of action of the wrench is, therefore,

$$\frac{x - \dfrac{aG \sin \theta}{R}}{l} = \frac{y - \dfrac{bG \sin \theta}{R}}{m} = \frac{z - \dfrac{cG \sin \theta}{R}}{n}.$$

58. Conditions for a couple. In § 55 it is shown that any system of forces may be reduced to a principal force R and a principal couple G. Since the principal force R is invariable in magnitude and direction whatever point O is chosen for the base point, it follows that if $R = 0$ the resultant of the system must be the principal couple G unless $G = 0$.

Hence the conditions that any system of forces shall reduce to a couple are

$$R = 0 \quad \text{and} \quad G \neq 0.$$

59. Conditions for a single force. If the principal couple $G = 0$, it follows that the resultant is a single force R unless $R = 0$. However, it is not necessary that $G = 0$ in order that the resultant of the system be a single force. For in § 56 it is shown that the component couple $G \sin \theta$ and the principal force R reduce to a single force $R' = R$. Therefore if the component couple $G \cos \theta = 0$, the final resultant is a single force $R' = R$.

Hence the conditions that a system of forces shall reduce to a single force are

$$G \cos \theta = 0 \quad \text{and} \quad R \neq 0.$$

Since $\cos \theta = ll' + mm' + nn'$, and $l = \frac{X}{R}$, $m = \frac{Y}{R}$, $n = \frac{Z}{R}$, $l' = \frac{L}{G}$, $m' = \frac{M}{G}$, and $n' = \frac{N}{G}$, the conditions for a single resultant force become

$$LX + MY + NZ = 0 \quad \text{and} \quad R \neq 0.$$

60. Conditions of equilibrium. In order that a body may be in equilibrium it is necessary that the principal force R and the principal couple G shall separately vanish. This necessitates that

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0$$

and
$$L = 0, \quad M = 0, \quad N = 0$$

for any base point.

By choosing different base points and different directions for the axes an infinite number of equations of equilibrium may be obtained. Only six of these equations, however, are independent.

EXAMPLES

1. Forces of 8 lb. and 12 lb. are at right angles to each other and 1.5 ft. apart, as shown in Fig. 129. Find (*a*) the resultant force and the resultant couple, (*b*) the resultant wrench.

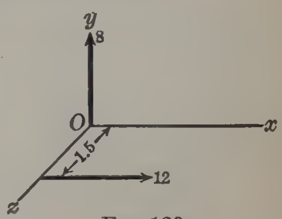

FIG. 129

Solution. (*a*) The base point O and the axes may be selected as shown. Then

$$\Sigma X = 12 \text{ lb.},$$
$$\Sigma Y = 8 \text{ lb.},$$
$$\Sigma Z = 0 \text{ lb.},$$

from which
$$R = \sqrt{208} = 14.42 \text{ lb.}$$

The direction cosines of the resultant R are

$$l = \frac{12}{\sqrt{208}} = 0.832, \quad m = \frac{8}{\sqrt{208}} = 0.554, \quad n = 0.$$

The moments about the axes are

$$L = 0 \text{ lb.-ft.}, \quad M = 18 \text{ lb.-ft.}, \quad N = 0 \text{ lb.-ft.},$$

and the resultant moment is
$$G = 18 \text{ lb.-ft.}$$

The direction cosines of the plane of the resultant couple G are

$$l' = 0, \quad m' = 1, \quad n' = 0.$$

The direction cosines of any plane are the direction cosines of its normal, and hence l', m', and n' are the direction cosines of the axis of the couple G.

(*b*) Since the couple can be moved to any parallel position, its vector or axis may be moved so as to pass through O. From solid analytic geometry, the angle θ between G and R is given by the equation

$$\cos \theta = ll' + mm' + nn'.$$

Hence
$$\cos \theta = \frac{8}{\sqrt{208}} = 0.554.$$

The component of G whose axis coincides with R is

$$G \cos \theta = 18 \times 0.554 = 9.98 \text{ lb.-ft.}$$

The component couple of G whose axis is perpendicular to R and whose plane lies in the plane determined by R and G is

$$G \sin \theta = 18 \times 0.832 = 14.98 \text{ lb.-ft.}$$

In order that each force in this couple may be R, the arm of the couple must be

$$\frac{G \sin \theta}{R} = \frac{14.98}{14.42} = 1.039 \text{ ft.}$$

This component couple, $G \sin \theta$, may be combined with R so that one of the forces, R, of the couple annuls the original resultant R, leaving a single force R, parallel to the original resultant R, lying in the plane determined by R and G and at a distance from R equal to $\frac{G \sin \theta}{R}$. The equation of the plane determined by R and G is

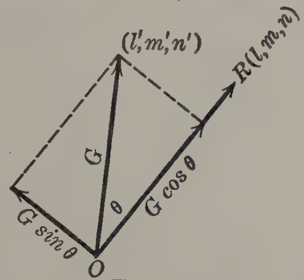

FIG. 130

$$ax + by + cz = 0,$$

where a, b, c are the direction cosines of the normal to the plane. These are determined from the equations

$$\frac{a}{mn' - m'n} = \frac{b}{l'n - ln'} = \frac{c}{lm' - l'm}$$

and $$a^2 + b^2 + c^2 = 1.$$

In this case $$\frac{a}{0} = \frac{b}{0} = \frac{c}{0.832},$$

from which $$a = 0, \quad b = 0,$$

and from $a^2 + b^2 + c^2 = 1$ it follows that $c = 1$. The equation of the plane of R and G is therefore $$z = 0.$$

The final resultant wrench is a force $R = 14.42$ lb. acting along the line whose direction cosines are $l = 0.832$, $m = 0.554$, $n = 0$ and which passes through the point (0, 0, 1.039) together with a couple whose moment is $G \cos \theta = 9.98$ lb.-ft. and whose axis coincides with R or is parallel to R.

2. A uniform bar of length d and weight W rests with its lower end A in the corner of a box of width b and height c. The upper end B of the bar rests against a side of the box and is prevented from sliding down by a string of length d' attached to the upper corner C of the box. Find the tension in the string and the pressure against the box.

Solution. The forces acting upon the bar are W, T, B_z, A_x, A_y, and A_z. Applying the conditions of equilibrium (Fig. 131),

$$\Sigma X = 0 \text{ gives } A_x - T \cos \theta = 0, \tag{1}$$

$$\Sigma Y = 0 \text{ gives } A_y + T \sin \theta - W = 0, \tag{2}$$

$$\Sigma Z = 0 \text{ gives } A_z - B_z = 0, \tag{3}$$

$$L = 0 \text{ gives } B_z(c - d' \sin \theta) + W\frac{d}{2} - A_z b = 0, \tag{4}$$

$$M = 0 \text{ gives } A_x b - B_z d' \cos \theta = 0, \tag{5}$$

$$N = 0 \text{ gives } Tc \cos \theta - \frac{Wd' \cos \theta}{2} = 0. \tag{6}$$

From (6),
$$T = \frac{Wd'}{2c}.$$

From (1),
$$A_x = \frac{Wd' \cos\theta}{2c}.$$

From (2),
$$A_y = \frac{W(2c - d' \sin\theta)}{2c}.$$

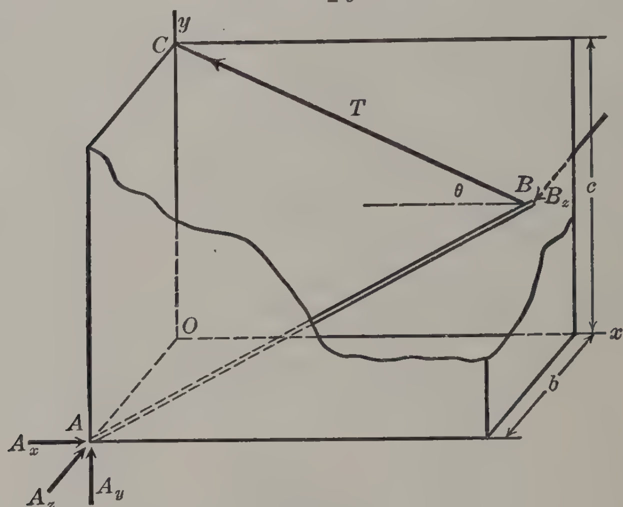

FIG. 131

From (3) and (5),
$$B_z = \frac{Wb}{2c} = A_z.$$

From the geometry of the figure,
$$\sin\theta = \frac{b^2 + c^2 - d^2 + d'^2}{2cd'},$$
by means of which A_x and A_y may be expressed in terms of the given dimensions.

PROBLEMS

1. Resolve a force of 100 lb. into three rectangular components whose magnitudes are proportional to the numbers 3, 4, and 5.

Ans. 42.43 lb., 56.57 lb., 70.71 lb.

2. A rectangular parallelepiped is inscribed in a sphere. From one vertex of the parallelepiped three forces, each represented by the length of an edge, act along the edges. Show that their resultant is represented by the diameter of the sphere through the vertex.

3. Three equal forces P act along three nonparallel nonintersecting edges of a rectangular parallelepiped. What must be the relation between the lengths a, b, c of the edges so that the resultant may be a single force? *Ans.* One edge is the sum of the other two edges.

4. Given two nonintersecting forces in space, MN and PQ in Fig. 132. Show that their resultant in *magnitude* and *direction* is $2\,AB$, where A and B are the mid-points of MQ and NP respectively.

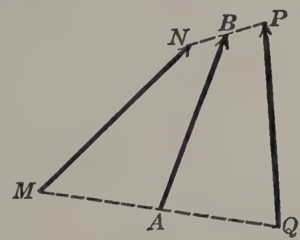

FIG. 132

5. Six equal forces P act along the edges of a regular tetrahedron as shown in Fig. 133. Find the resultant wrench.

Ans. $G = \dfrac{Pa\sqrt{3}}{2}$; $R = P\sqrt{6}$ through A, perpendicular to plane BCD.

6. The bar MN is subjected to forces of 40 lb., 60 lb., and 100 lb. lying in the plane of the paper and to a couple of 300 lb.-ft., as shown in Fig. 134. Find the resultant.

Ans. A couple whose moment = 500 lb.-ft. Its axis makes an angle $\theta = \text{arc tan}\left(-\frac{4}{3}\right)$ with MN and lies in a plane perpendicular to the paper through MN.

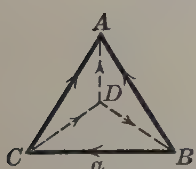

FIG. 133

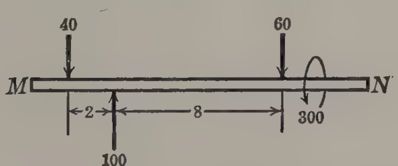

FIG. 134

7. A uniform bar AB, of length l, is constrained to rotate, Fig. 135, about its middle point O in a horizontal plane. A string attached at B passes over a pulley C and sustains a weight W. The pulley C is at a height h above D, the neutral position of the end B of the rod. Find the force P in the horizontal plane such that the bar AB may make a given angle θ with the line ED. For what value of θ is P a maximum?

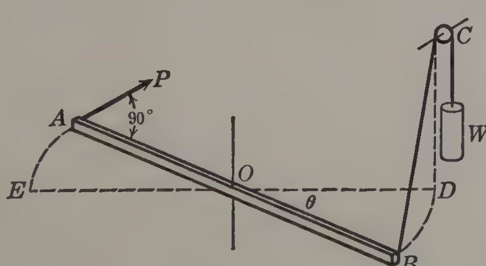

FIG. 135

$$Ans.\quad P = \frac{W}{2}\,\frac{l \sin\theta}{\sqrt{h^2 + l^2 \sin^2\dfrac{\theta}{2}}},\quad \tan\frac{\theta}{2} = \sqrt[4]{\frac{h^2}{l^2 + h^2}}.$$

8. Six uniform bars each of length l and weight W are joined together at their ends to form a regular tetrahedron which rests on a smooth horizontal plane. Find the tension in each bar of the base.

Ans. $\dfrac{W}{2\sqrt{6}}$.

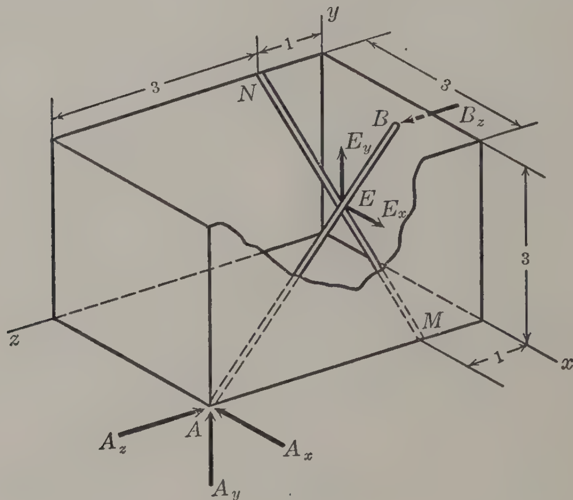

FIG. 136

9. A rod MN is fixed in a rectangular box as shown in Fig. 136. A rod 5 ft. long weighing 30 lb. is placed in the box so as to rest on the rod MN at E, one end of the rod being in the corner A and the other end resting against a side at B. Find the rectangular components of the force on the rod at A, also the force at E and at B.

Ans. $A_x = 10$ lb., $A_y = 20$ lb.,
$A_z = B_z = E = 10\sqrt{2}$ lb.

CHAPTER VIII

GRAPHICAL STATICS

61. Introduction. The method used in previous chapters to reduce any system of forces to an equivalent system and also to establish the conditions of equilibrium is known as the analytic method. There are, however, other methods of obtaining these results and also other ways of stating them. Two important methods are known as the *graphical method* and the method of *virtual velocity* or *virtual work*. While any method may be applicable, yet each method has its especial field of usefulness. In problems concerning frameworks, such as bridges and roofs, graphical processes have the advantage of elegance and simplicity and are much used by engineers. Graphical methods are especially advantageous in determining the stresses in complicated frameworks under unsymmetrical loads, since in such cases the computation by the analytic method is long and tedious. Moreover, the graphical method has the further advantages of affording a ready check upon the accuracy of the work and of clearly picturing the relations among the stresses.

62. Space diagram; force diagram or polygon; Bow's notation. Two essential elements or figures used in the graphical method are called the *space diagram* and the *force diagram*. The *space diagram* shows the location or position and direction of the lines of action of the known forces but not necessarily the magnitude of the forces. The *force diagram* shows the direction and magnitude of the forces but not their actual location or position.

The reading of the diagrams is facilitated by a notation introduced by Mr. R. H. Bow. In his notation a force in the space diagram is designated and named by two *capital* letters placed one on each side of the line of action of the force. The corresponding force in the force diagram or polygon is designated by the same two *small* letters placed one at each end of the vector representing the force.

For example, in the frame shown in Fig. 137, the spaces between the external forces and also the compartments of the frame are each designated by a capital letter. The lines of action of the external forces are designated by AB, BC, and CA. The lines of action of stresses in the members of the frame are represented by BD, DA, ED, EC, and AE.

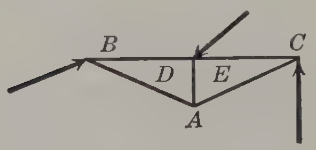

FIG. 137. Space diagram

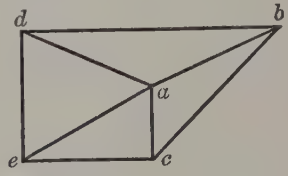

FIG. 138. Force diagram

In the force diagram (Fig. 138) the magnitudes of the various forces are designated by the corresponding small letters. Thus the magnitude of the stress in DA, Fig. 137, is da in Fig. 138.

63. Concurrent forces. Given the system of concurrent forces AB, BC, and CD, acting at the point O of any body as shown in the space diagram (Fig. 139); to find the resultant and equilibrant

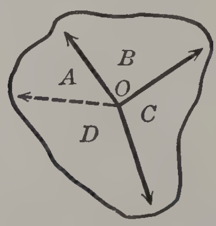

FIG. 139. Space diagram

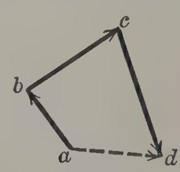

FIG. 140. Force diagram

of the system. The force diagram, or force polygon (Fig. 140) is constructed by laying off to scale, from any point a, the forces ab, bc, and cd as in § 18. The *resultant* of the system is represented in magnitude and direction by the closing line ad. Its point of application is O. The equilibrant of the system is the force da in the force polygon or the force DA in the space diagram. It is evident that the condition for the equilibrium of a system of *concurrent* forces is that *the force polygon must close.*

64. Resultant of nonconcurrent coplanar forces; string polygon. Given the system of nonconcurrent forces acting on any body as represented by P_1, P_2, P_3, and P_4 in the space diagram,

Fig. 141, to find the resultant and equilibrant of the system. In Bow's notation the forces P_1, P_2, P_3, and P_4 are designated by AB, BC, CD, and DE respectively.

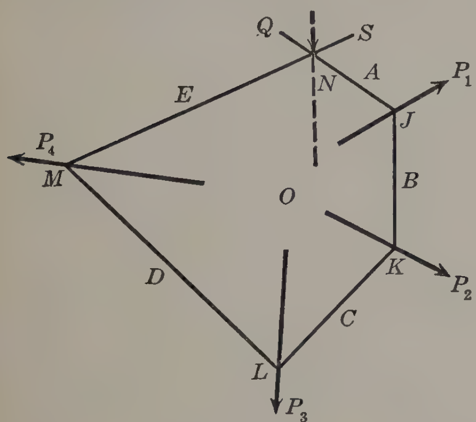

FIG. 141

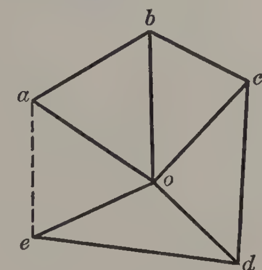

FIG. 142

The force polygon is constructed by drawing to scale, from any point a,

ab parallel and equal to P_1,
bc parallel and equal to P_2,
cd parallel and equal to P_3,
de parallel and equal to P_4.

Then ae is the resultant in *magnitude* and *direction* of the forces P_1, P_2, P_3, and P_4.

The resultant will be completely determined when a point on its line of action has been found. In order to determine such a point it is necessary to construct another polygon called the *string polygon* or funicular polygon.

Join the vertices of the force polygon to any point o, called the pole. The rays oa, ob, etc. may be considered as forces. Thus ao and ob have ab for their resultant; bo and oc have bc for their resultant, and so on; the four forces P_1, P_2, P_3, and P_4 being replaced by eight forces ao, ob, bo, oc, co, od, do, oe.

Through any point J on P_1 draw

JQ (or AO) parallel to ao,
JK (or BO) parallel to bo,
KL (or CO) parallel to co,
LM (or DO) parallel to do,
MS (or EO) parallel to eo.

The intersection of the first side JQ and the last side MS of this polygon determines a point N on the line of action of the resultant force. The figure $NJKLMN$ is called a string polygon.

The following considerations show that N must be a point on the line of action of the resultant. The force P_1 acting at J may be resolved into two component forces acting along NJ and KJ. These component forces are represented in the force polygon by ao and ob, and their resultant is ab or P_1, as shown above. Similarly, P_2 may be resolved into the forces bo and oc acting along JK and LK, and so on. Thus the original forces P_1, P_2, P_3, and P_4 are replaced by the forces ao, ob, bo, oc, co, od, do, oe, and these forces act along the sides of the string polygon. Furthermore, since bo and ob annul each other, likewise co and oc, and od and do, there remain ao acting along NJ and oe acting along NM. The resultant of these two forces is ae in the force polygon, and its line of action passes through the point N in the string polygon.

Finally, the resultant of the nonconcurrent forces P_1, P_2, P_3, and P_4 is a force whose *magnitude* and *direction* are represented by ae, the closing line of the force polygon drawn from the first point to the last point, and whose line of action is parallel to ae, passing through the point of intersection N of the first and last sides of the string polygon. The equilibrant is equal and opposite to the resultant.

For each selection of the pole or starting point of the string polygon there will be a different string polygon. The intersections of the first and last sides of each of these string polygons, however, all lie on a straight line, that is, the line of action of the resultant force. The pole should be chosen so that the sides of the string polygon will not intersect at very acute angles or outside of the drawing.

EXAMPLE

Find the resultant of the following system of forces:

$$P_1 = 200 \text{ lb. at } (2, 2),\ 30°,$$
$$P_2 = 150 \text{ lb. at } (3, -1),\ -45°,$$
$$P_3 = 175 \text{ lb. at } (0, -3),\ -105°,$$
$$P_4 = 225 \text{ lb. at } (-1, 1),\ 150°.$$

Solution. Graphical method.

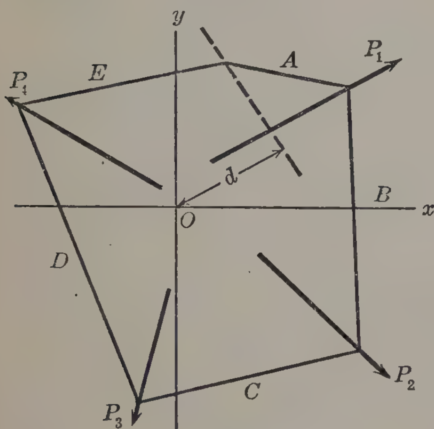

FIG. 143

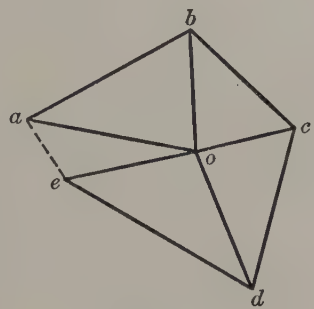

FIG. 144

Analytic method.

$$\begin{aligned}
\Sigma X = 200\cos 30^\circ &= 173.21 \\
150\cos 45^\circ &= 106.05 \\
-175\cos 75^\circ &= -45.29 \\
-225\cos 30^\circ &= \underline{-194.85} \\
&\quad +39.1
\end{aligned}
\qquad
\begin{aligned}
\Sigma Y = 200\sin 30^\circ &= 100.00 \\
-150\sin 45^\circ &= -106.05 \\
-175\sin 75^\circ &= -169.03 \\
225\sin 30^\circ &= \underline{112.50} \\
&\quad -62.6
\end{aligned}$$

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2} = 73.8 \text{ lb.}$$

$$\theta = \tan^{-1}\frac{\Sigma Y}{\Sigma X} = \tan^{-1}(-1.60) = 122^\circ.$$

COUPLES

NEGATIVE		POSITIVE	
$173.21 \times 2 =$	346.42	$100.00 \times 2 =$	200.00
$106.05 \times 3 =$	318.15	$106.05 \times 1 =$	106.05
$45.29 \times 3 =$	135.87	$169.03 \times 0 =$	0.00
$112.50 \times 1 =$	112.50	$194.85 \times 1 =$	194.85
	-912.9		$+500.9$
	$+500.9$		

$$G = -412.0 \text{ lb.-ft.}$$

$$d = \frac{G}{R} = \frac{412.0}{73.8} = 5.58 \text{ ft.}$$

RESULTS

GRAPHICAL	ALGEBRAIC
$R = 74.0$ lb.	$R = 73.8$ lb.
$\tan\theta = -1.63$	$\tan\theta = -1.60$
$d = 5.63$ ft.	$d = 5.58$ ft.

PROBLEM

Determine, by means of the string and force polygons, the magnitude and direction of the resultant of the system of forces shown in Fig. 145. Determine also the point of intersection of the line of action of the resultant with the x axis.

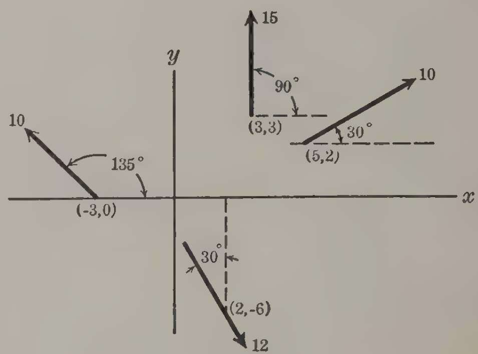

FIG. 145

65. The conditions for equilibrium of nonconcurrent coplanar forces. In § 64 it is shown that the system of nonconcurrent forces P_1, P_2, P_3, and P_4 reduces to two forces, represented in the force polygon by *ao* and *oe*. Their lines of action are the first and last sides of the string polygon.

The necessary and sufficient conditions for the equilibrium of these two forces or any two forces is that they must be equal in magnitude and opposite in direction and have the same line of action.

In order that the forces *ao* and *oe* may be equal and opposite, the points *a* and *e* must coincide; that is, *the force polygon must close.* In order that these forces may have the same line of action, *NM* and *NJ* must coincide; that is, *the string polygon must close.*

Hence *the conditions of equilibrium of any system of nonconcurrent coplanar forces is that the force polygon must close and the string polygon must close.*

The fact that the force polygon closes shows that the resultant force is zero, although the original system of forces may reduce to a couple. The system of nonconcurrent forces reduces to a couple when the force polygon closes and the first and last sides of the string polygon are parallel. When these parallel sides coincide, the system is in equilibrium. If the force polygon does not close, the resultant of the system is a resultant force and no couple.

Since the equal and opposite forces may be considered as existing in the separate strings of a string polygon, it is evident

that a system of bars connected by smooth pins and having the shape of the string polygon will be in equilibrium under the action of the external forces applied at the joints. The string polygon is sometimes known as the link polygon or equilibrium polygon.

The general method of the solution of problems in equilibrium is to draw both the force polygon and the string polygon as far as possible from the data, and then complete them subject to the condition that they must each close.

66. Parallel forces; single resultant or equilibrant. When all the forces are parallel the force polygon collapses into a straight line. In Fig. 146 forces AB, BC, CD, and DE are given and it is

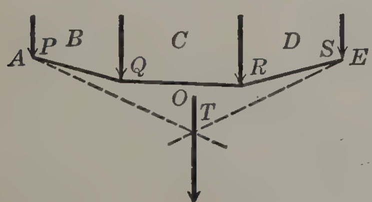

FIG. 146

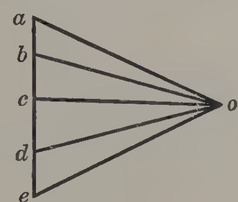

FIG. 147

required to find their resultant. The force polygon shown in Fig. 147 gives the magnitude and direction of the resultant force by the vector ae. The equilibrant is the vector ea. From any point P the string polygon is drawn, having strings parallel to ao, bo, co, etc. One point T on the line of action of the resultant force ae is determined by the intersection of the first and the last strings, OA and OE. The resultant of the parallel-force system shown is therefore a force which is given in magnitude and direction by the vector ae and which passes through the point T. The equilibrant is equal and opposite to the resultant.

67. Parallel forces; two equilibrants. It is sometimes necessary to find *two* equilibrants of a system of parallel forces, as, for example, in the case of a beam supported at any two points and carrying a number of loads.

Let AB, BC, and CD be the loads acting upon a beam MN, and let it be required to determine two parallel equilibrants of this system of forces through the points M and N. After choosing a pole o and drawing the rays oa, ob, oc, and od, draw

the sides of the string polygon in the usual way. Let m and n be the points where the strings OA and OD meet the lines of action of the equilibrants through M and N. Let the points m and n be joined by a straight line, and let a line be drawn from o parallel to mn to intersect the force polygon $abcd$ at e. The magnitude and direction of the equilibrant at M acting

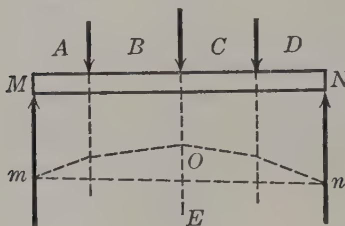

FIG. 148

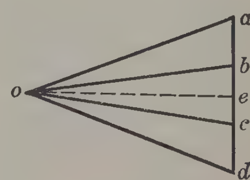

FIG. 149

along EA are given by the vector ea. Likewise the equilibrant acting along DE is given by de. The proof of this construction is left to the student.

PROBLEMS

1. A horizontal beam 20 ft. long resting on supports at its ends carries a uniformly distributed load of 100 lb. per foot and concentrated vertical loads of 1 T., 2 T., 4 T., and 6 T. at distances of 5 ft., 8 ft., 12 ft., and 16 ft. respectively from the left support. Determine the reactions at the supports by the graphical method and check the results. *Ans.* 5.25 T., 8.75 T.

2. A horizontal beam AB 18 ft. long rests upon a smooth hinge at A and upon a roller at B. Forces of 100 lb., 200 lb., 300 lb., and 400 lb. act upon the beam at points 3 ft., 6 ft., 10 ft., and 14 ft. from the left end A, their lines of action making angles of 135°, 120°, 90°, and 30°, respectively, with AB. The angles are laid off in the usual trigonometric sense. Determine by graphical methods the magnitude and direction of the reactions at A and B.

Ans. $R_A = 403.4$ lb., at 64° 10′; $R_B = 392.0$ lb.

68. String polygon through two given points. Let AB, BC, and CD, Fig. 150, be the given forces, and let it be required to pass a string polygon through the points m and n. Draw a line np through n parallel to the closing line ad of the force polygon (Fig. 151). Select any pole o', and draw the string polygon for the pole o', beginning at one of the points, m, through which

the string polygon is required to pass, and suppose that the last side of the string polygon intersects the line np in q. Draw a line through o' parallel to mq to intersect the line ad in e. It is clear from § 67 that de and ea represent the equilibrants which, applied at n and m respectively, keep the forces AB, BC, and CD in equilibrium. It is also evident that the position of

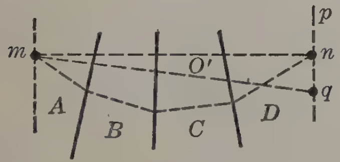

FIG. 150

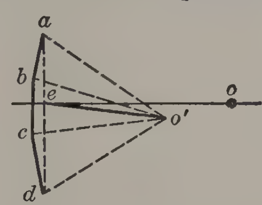

FIG. 151

the point e which determines the magnitude of the equilibrants is independent of the position of the pole o'. Since the position of e is invariable and since the line joining e to the pole is parallel to the closing side mq of the string polygon, the pole required to pass the string polygon through the point n must lie upon a line through e parallel to the closing line mn of the required string polygon. Hence to complete the construction, any point o on a line eo through e parallel to mn is taken as a pole. The corresponding string polygon will pass through the points m and n. There are an infinite number of string polygons through the given points.

69. String polygon through three given points. This construction is of importance in connection with the equilibrium of the three-hinged arch (§ 76).

Let AB, BC, CD, DE, and EF, Fig. 152, be the given forces, and let it be required to pass the string polygon through the three points m, n, and p. Construct the force polygon $abcdef$; choose any point o' as pole, and draw the string polygon, starting at one of the given points n. Pass a side through n parallel to the ray $o'c$. Draw the closing lines ca for the forces to the left, and fc for the forces to the right, of the point n. Draw the action line of the force ca through the point m and the action line of the force fc through the point p, parallel to ca and fc respectively. Let S and T denote the intersection points of the string polygon with the action lines of the equilibrants ca and fc respectively. Through o' draw lines parallel to nS and nT to intersect ca

and fc at x and y. From § 68 it is evident that the poles for the string polygon through n and m will lie on a line through x parallel to mn and likewise that the poles for the string polygon passing through n and p will lie on a line through y parallel to np. The pole necessary to pass the string polygon through the points m, n, and p therefore lies at the intersection o of the two last-named lines. The required string polygon is not shown in Fig. 152. A check or lack of check upon the accuracy of the

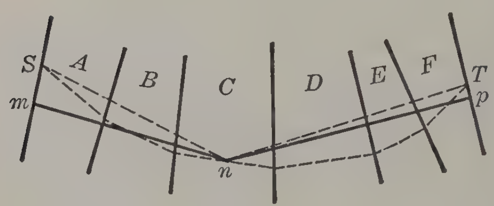

FIG. 152

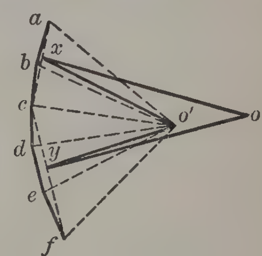

FIG. 153

determination of the point o is afforded by the circumstance of the string polygon's actually passing through the three given points or its failure to pass through them.

70. Stresses in frames. The graphical or analytic method of finding the stresses in determinate frames (that is, frames which do not contain more than the requisite number of members to make them rigid) may be briefly stated as follows: (1) The reactions on the frame are found by regarding the whole frame as a rigid body in equilibrium under the action of a system of coplanar forces which are usually nonconcurrent. (2) If the joints are pin-connected, the forces which act at a joint form a system of concurrent forces in equilibrium. If the joints are held together with rivets, the system of forces may usually be considered as concurrent without serious error. (3) Any portion of the frame may be separated from the remainder and considered as a rigid body in equilibrium under the action of the loads acting upon it, together with the forces which are applied to it through the action of the members, joints, etc., of the remainder of the frame. A particular case under this method is known as the *method of sections.*

71. The method of joints; graphical solution. The forces which support a loaded frame or truss are called *reactions.* In

many cases there are only two reactions. As a preliminary to either the method of joints or the method of sections, the reactions must be determined by § 67 or otherwise. The method of joints, considered from either the graphical or the analytic standpoint, consists in the solution of as many systems of concurrent forces as there are joints. After the reactions have been determined the work is begun at a joint where not more than two unknown forces meet with any number of known forces.

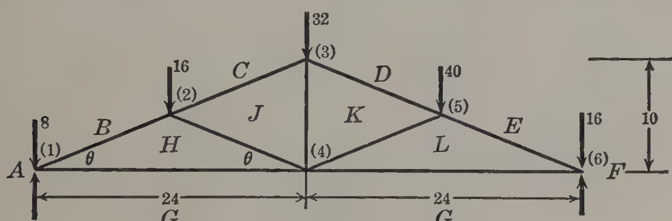

FIG. 154

After the unknown forces are determined at this joint, an adjacent joint is selected and the same process is repeated, and so on until all the joints where unknown forces exist have

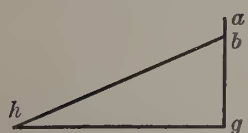

FIG. 155

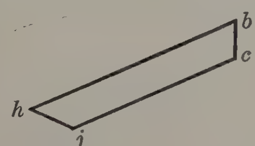

FIG. 156

FIG. 157

been considered. The method of joints will be best understood by considering the solution of a problem by both graphical and analytic methods.

Let the simple truss shown in Fig. 154 be acted upon by the forces AB, BC, CD, DE, and EF. When the impressed forces are parallel it is easier to determine the reactions by the analytic method. The reaction FG is 66 pounds and the reaction GA is 46 pounds. In accordance with the usual convention the forces at the left-hand joint are first considered.

Proceeding around the joint in the clockwise direction, the force GA is laid off to scale as ga, Fig. 155, and the force AB as ab. A line is drawn through b parallel to the line of action of the force BH, and a second line is drawn through g parallel to

HG to intersect it at *h*. The forces *BH* and *HG* may now be scaled off from the force polygon as *bh* and *hg*. Proceeding to adjacent joints where there are no more than two unknown forces, there is obtained the series of polygons (Figs. 156 to 160 inclusive) from which all the unknown forces may be scaled. It is important to notice that the directions as well as the magnitudes of the unknown forces are given by going around the force polygons in the same sense as the known forces. Thus in Fig. 155 the order of reading the forces around the first joint is

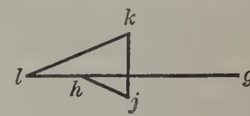

FIG. 158

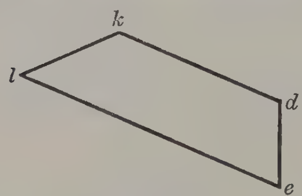

FIG. 159

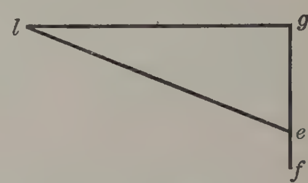

FIG. 160

ga, *ab*, *bh*, *hg*. The corresponding order in the space diagram of Fig. 154 is *GA*, *AB*, *BH*, *HG*. Proceeding around the force polygon in the order named, the force *bh* acts toward the joint (1) and therefore the member *BH* is in compression. The force *hg* acts away from the joint (1) and hence the member *HG* is in tension. On account of the inconvenience of drawing separate force diagrams for each joint, the force diagrams are always combined into a single stress diagram as shown in Fig. 161. In Fig. 161 each of the sides of the polygon taken in opposite directions represents two separate forces, that is, a side in each of the separate force polygons for the individual joints. Thus *bh* in Fig. 161 represents both *bh* in Fig. 155 and *hb* in Fig. 156. In order to determine whether the members *DK* and *KJ* are in tension or in compression, the forces acting at the joint (3), Fig. 154, are read in the clockwise order, *CD*, *DK*, *KJ*, *JC*. Following the same order in Fig. 161, the force *cd* acts

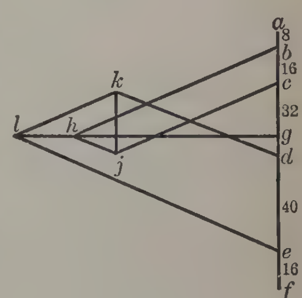

FIG. 161

toward the joint (3), *dk* acts toward the joint, *kj* acts away from the joint, and *jc* acts toward the joint. Hence the members *DK* and *JC* are in compression and the member *KJ* is in tension. If the forces acting at the joint (3), Fig. 154, had been read in the counterclockwise order, the force *dc* in Fig. 161 would have acted upward away from joint (3), which is contrary to the actual condition.

72. The method of joints; analytic solution. The analytic method may be illustrated by the solution of the example of § 71. From the dimensions of the truss it is evident that a triangle whose sides are in the ratio of 5, 12, and 13 will furnish the required geometric relations. Let it be assumed that the forces *BH* and *HG* are both tensions. Then at the left-hand joint

$$\Sigma X = 0 \text{ gives } \tfrac{12}{13}\,BH + HG = 0, \tag{1}$$

and

$$\Sigma Y = 0 \text{ gives } 46 - 8 + \tfrac{5}{13}\,BH = 0. \tag{2}$$

From (2), $BH = -\,98.8 \text{ lb.}$

and from (1), $HG = -\tfrac{12}{13}\,BH,$ or $HG = 91.2 \text{ lb.}$

The negative sign before *BH* shows that an incorrect assumption was made in regard to the nature of the stress; the force acts toward the joint and hence the member *BH* is in compression. The fact that the force *HG* is positive in sign shows that the member *HG* is in tension, as assumed. The signs plus and minus as derived from these equations should not be confused with the conventional signs (nonalgebraic) which are sometimes prefixed to the magnitude of the force to show whether the force is tension or compression. According to the convention the plus sign indicates tension and the negative sign indicates compression.

Proceeding to an adjacent joint where there are not more than two unknown forces acting (joint (2)),

$$\Sigma X = 0 \text{ gives } \tfrac{12}{13} \times 98.8 - \tfrac{12}{13}\,CJ - \tfrac{12}{13}\,JH = 0, \tag{3}$$

and

$$\Sigma Y = 0 \text{ gives } \tfrac{5}{13} \times 98.8 + \tfrac{5}{13}\,JH - \tfrac{5}{13}\,CJ - 16 = 0, \tag{4}$$

where both the unknown forces are assumed to act toward the joint. Solving (3) and (4),

$$CJ = 78 \text{ lb. and } JH = 20.8 \text{ lb.}$$

Since both signs are plus the forces are compressive, as assumed.

Let the top joint be considered. Since the stress in CJ was found to be compression, it acts toward the joint, and

$$\Sigma X = 0 \text{ gives } \tfrac{12}{13} \times 78 - \tfrac{12}{13}\, DK = 0, \tag{5}$$

also $\quad \Sigma Y = 0$ gives $-32 + JK + \tfrac{5}{13} \times 78 + \tfrac{5}{13}\, DK = 0. \qquad (6)$

From (5), $\qquad DK = 78$ lb. compression.

From (6), $\qquad JK = -28$ lb.

The minus sign shows that the stress in JK was wrongly taken to be compression; it is tension.

Proceeding to joint (4),

$$\Sigma X = 0 \text{ gives } -91.2 + \tfrac{12}{13} \times 20.8 - \tfrac{12}{13}\, KL + LG = 0, \tag{7}$$

also $\Sigma Y = 0$ gives $-\tfrac{5}{13} \times 20.8 + 28 - \tfrac{5}{13}\, KL = 0. \qquad (8)$

From (8), $\qquad KL = 52$ lb. compression

and from (7), $\qquad LG = 120$ lb. tension.

At joint (5)

$$\Sigma X = 0 \text{ gives } \tfrac{12}{13} \times 52 + \tfrac{12}{13} \times 78 - \tfrac{12}{13}\, EL = 0, \tag{9}$$

from which $\qquad EL = 130$ lb. compression.

As a check, the forces EL and LG may be obtained independently by writing the equations of equilibrium for the forces acting at joint (6).

73. The method of sections. The method of joints is especially useful when the loading remains constant. If, however, the loading is not constant, as, for example, in the case of a bridge supporting a moving train, the stresses in the members will vary depending upon the position of the loads. When the stress in a single member or a group of members is desired for any given loading, it is usual to make use of the method of sections. The analytic solution is generally employed on account of its brevity. The method of sections possesses the advantage of permitting the calculation of the stress in a member near the middle of a truss without previously calculating the stresses in a large number of members.

The method consists essentially in supposing that the frame is cut into two portions by a section which passes through the members in which the stresses are required, and that the real frame on one side of the section has vanished and the remain-

ing portion of the frame is held in equilibrium by the loads, the reaction, and the forces applied at the section which would ordinarily exist in the cut members were the frame a complete whole. The section must not cut more than three nonconcurrent nonparallel members, since not more than three independent equations of equilibrium may be written. An example will illustrate the procedure to be followed.

Let it be required to find the stress in the members CJ, JH, and HG, Fig. 154, § 71. Pass a section through the truss, cutting the members in which the stress is required, and remove the part of the truss at the right of the section, as shown in Fig. 162. Let forces T_1, T_2, and T_3 act at the ends of the severed members, and let the required forces be assumed to be tensions. To find the force T_1 it is convenient to take moments about the intersection B of the action lines of the unknown forces T_2 and T_3. This choice of moment center is made to avoid the introduction of more than one unknown force into the equation.

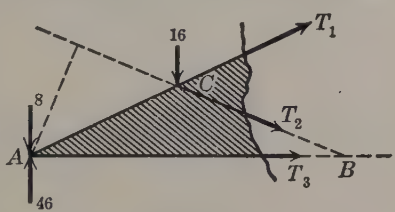

FIG. 162

$$\Sigma M_B = 0 \text{ gives } 8 \times 24 - 46 \times 24 + 16 \times 12 - 24 \times \tfrac{5}{13}\, T_1 = 0,$$

from which $$T_1 = -\,78 \text{ lb.}$$

The minus sign indicates that the force is compression, instead of tension as assumed.

To find the force T_3, moments are taken about the point C where the lines of action of T_1 and T_2 intersect.

$$\Sigma M_C = 0 \text{ gives } 8 \times 12 - 46 \times 12 + 5\, T_3 = 0,$$

from which $$T_3 = 91.2 \text{ lb. tension.}$$

To determine the force T_2, all the forces may be resolved horizontally or vertically or moments may be taken about the point A.

Thus $$\Sigma X = 0 \text{ gives } \tfrac{12}{13}\, T_1 + \tfrac{12}{13}\, T_2 + T_3 = 0,$$

and, since $T_1 = -\,78$ lb. and $T_3 = 91.2$ lb., therefore

$$-\tfrac{12}{13} \times 78 + \tfrac{12}{13}\, T_2 + 91.2 = 0,$$

from which $$T_2 = -\,20.8 \text{ lb.}$$

Or $\Sigma Y = 0$ gives $46 - 8 - 16 + \frac{5}{13}(-78) - \frac{5}{13}T_2 = 0,$

and $$T_2 = -20.8 \text{ lb.},$$

as before.

Likewise, $\Sigma M_A = 0$ gives $-16 \times 6 - \frac{5}{13} \times 12\, T_2 = 0,$

from which $$T_2 = -20.8 \text{ lb.}$$

The minus sign indicates that T_2 is compression.

74. The method of substitution. Certain types of structures appear to be indeterminate when in reality they are not. The Fink truss (Fig. 163) affords an example of such a structure.

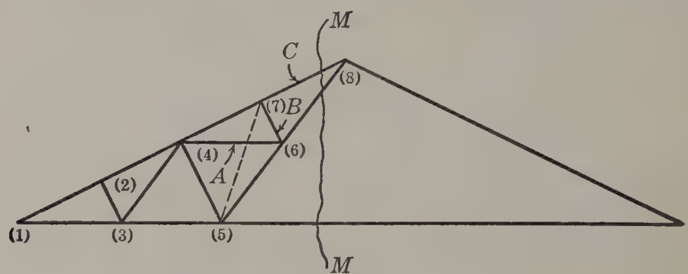

FIG. 163

If the stress diagram is constructed, a difficulty appears after the drawing has been completed for joints (1), (2), and (3), since more than *two* unknown concurrent forces act at both joints (4) and (5). One of the methods of avoiding this difficulty is to remove the members A and B and substitute for them the member shown by the dotted line. The polygons for joints (4) and (7) may be drawn to give the stress in the member C, since the stress in C is independent of the arrangement of the members in the section bounded by the joints (4), (5), (6), and (7). The original members are then replaced, after which the joints may be considered in the sequence (7)–(4)–(5)–(6). The stress in member C might have been determined by the method of sections (§ 73), taking the section M–M.

75. Statically indeterminate structures. Various methods have been devised for obtaining the stresses in frames having redundant members. The student will find indeterminate structures fully dealt with in treatises relating to the subject.

76. The three-hinged arch. The three-hinged arch has a hinge at each support and a third hinge located between the supports (Fig. 164). In this type of arch the line of thrust as given by the sides of the string polygon must pass through the hinges A, B, and C, since the hinges cannot resist a moment. The analytic solution of the problem is effected by writing the three equations of equilibrium for both sides of the arch.

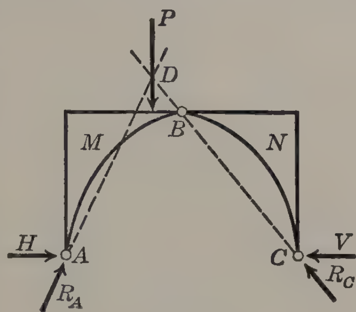

FIG. 164

The graphical solution. Let the arch (Fig. 164) support a single load. Since the weight of the arch is neglected, the side N does not support any load and is therefore in equilibrium under the action of two forces, one at B and the other at C. It follows that the line of action of the reaction at C must pass through B (§ 51).

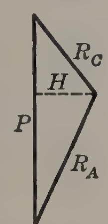

FIG. 165

Since three nonparallel coplanar forces which are in equilibrium must be concurrent (§ 52), it follows that the reaction at A must intersect the line of action of the load P at D, the point of intersection of the force R_C and the load P. The reactions at A and C are found from the force polygon (Fig. 165). The horizontal reaction H is the horizontal component of either R_A or R_C, and it is represented by a dotted line in the force polygon (Fig. 165). The graphical solution for several loads is effected by drawing a string polygon through the three given points A, B, and C (§ 69).

The reactions at A and C coincide with the strings through A and C of the string polygon which passes through A, B, and C. The magnitudes of the reactions are found by drawing two closing lines in the force polygon, parallel to the strings through A and C.

77. Loads not applied at the joints. The three-hinged arch is an example of a frame where the loads are not applied at the joints. An alternative method of solving problems of this kind is to replace the load upon each member by two component loads applied at the joints, that is, by two loads at the joints

which have a resultant equal to the given load, and then to draw the stress diagram in the usual way. The pin reactions are obtained by vectorial addition of the stresses found in the members with the component loads applied at the joints.

PROBLEMS

1. Draw the stress diagram for the pin-connected frame shown in Fig. 166.

2. Determine the stress in each member of the frame shown in Fig. 166 (*a*) by the method of sections, (*b*) by the method of joints. Also determine the vertical and horizontal components of the reactions of the wall.

3. Draw the stress diagram for the truss shown in Fig. 154, replacing the system of loads as follows: loads of 12 T. at joints (2), (3), (4), and (5) and loads of 6 T. at joints (1) and (6).

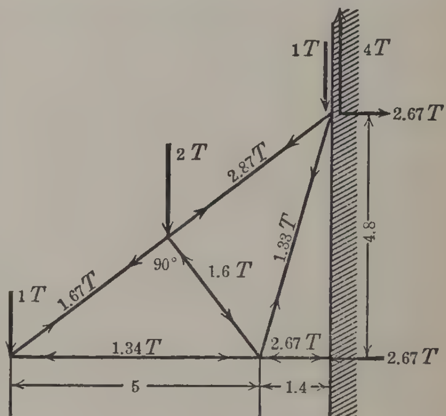

FIG. 166

4. Compute the stresses of Problem 3, (*a*) by the method of joints, (*b*) by the method of sections.

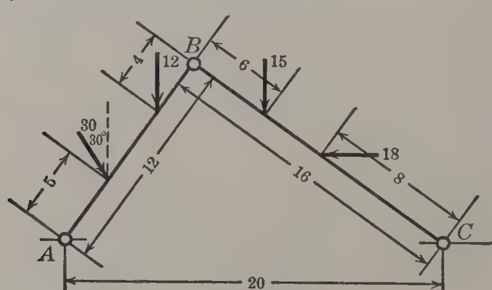

FIG. 167

5. Draw the string polygon for the forces acting upon the three-hinged arch shown in Fig. 167 through the points A, B, and C, and determine the reactions at A and C in magnitude and direction.

Ans. $R_A = 39.0$ lb., $81° 30'$; $R_C = 14.7$ lb., $79° 10'$.

6. Find the stresses in the chords A, B, C, D, and E of the Petit truss shown in Fig. 168. The two halves of the truss are symmetrically loaded.

Ans. $A = 1398$ T., $B = 1440$ T., $C = 1294$ T., $D = 1230$ T., $E = 90$ T.

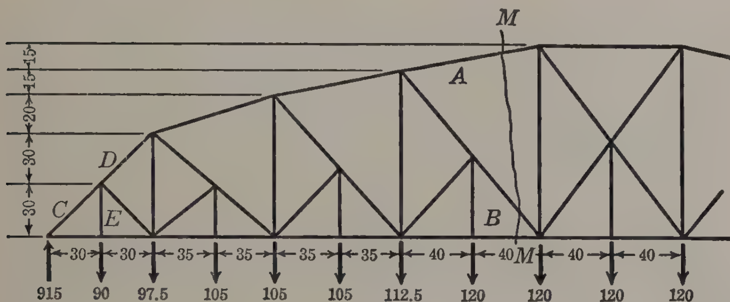

FIG. 168

7. Draw the stress diagram for the Fink truss shown in Fig. 169. The joints (1), (2), (3), (4), and (5) are equally spaced.

Ans. $BL = -18{,}200$, $MN = +2400$, $PQ = -1846$,
$LK = +16{,}800$, $NK = +14{,}400$, $OR = +4800$,
$CM = -17{,}431$, $ON = -3692$, $EQ = -15{,}892$,
$ML = -1846$, $PO = +2400$, $QR = +7200$,
$RK = +9600$, $DP = -16{,}662$.

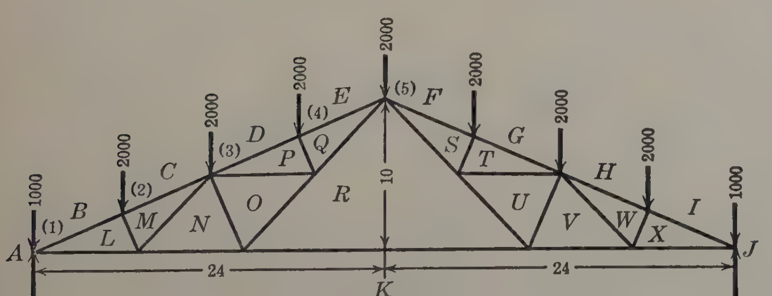

FIG. 169

8. Determine the stresses in each member of the Fink truss in Fig. 169 by the analytic method.

9. Draw a stress diagram for the truss shown in Fig. 170. The tension in the member GH is 1000 lb. Why? The stress in OP is zero. Why? Derive a general rule for the stresses in members which are connected at right angles; also a general rule for any three concurrent members two of which are collinear.

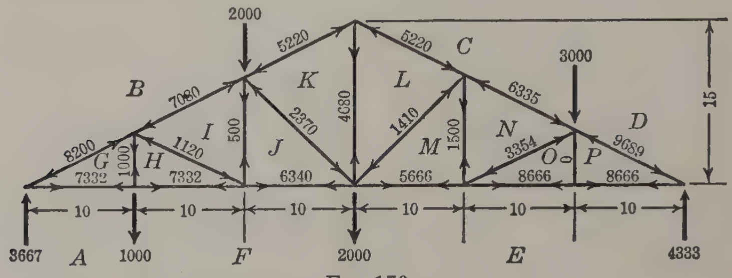

FIG. 170

10. Let the truss of Fig. 171 be acted upon by the wind loads shown. Find the reactions at A and B.

HINT. The resultant wind load of 4 T. acts at the joint (2). The reaction at A is vertical, since the truss is free to move horizontally upon the rollers shown. The truss is therefore in equilibrium under the action of only three forces, which must therefore be concurrent. The direction of the reaction at B is found by drawing a vertical line through A to meet the resultant load at C, and drawing the line BC. The line BC is evidently the line of action of the reaction at B. Construct the force triangle to determine the magnitude of the reactions at A and B.

Ans. $R_A = 1.95$ T.,
$R_B = 2.71$ T.,
$\theta = \tan^{-1} 0.521$.

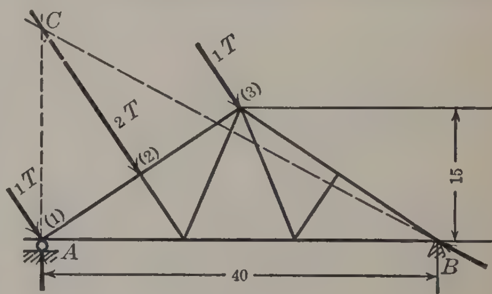

FIG. 171

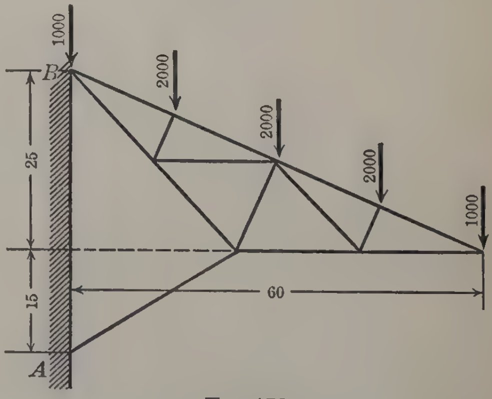

FIG. 172

11. Find the reactions at A and B graphically (Fig. 172).

12. The Fink truss of Fig. 173 is subjected to the wind loads shown. Show by *inspection* that the stresses in the members BC, CD, DH, HB, ED, EF, and FG are each zero. Also show by inspection that the stress in member 1–2 and in 5–6 is 10 T.; the

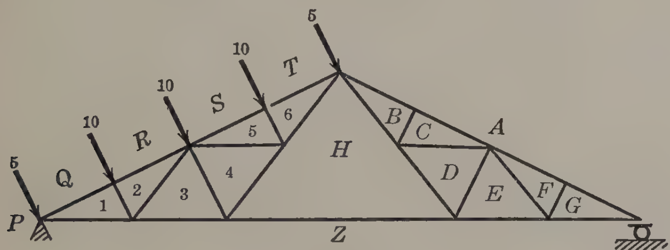

FIG. 173

stress in 3–4 is 20 T.; the stresses in the members Q–1, R–2, S–5, and T–6 are equal; the stress in 5–4 plus the stress in 4–H equals the stress in 6–H; the stress in 4–H plus the stress in H–Z equals the stress in 3–Z; the stress in 2–3 plus the stress in 3–Z equals the stress in 1–Z; and the stress in 2–3 equals the stress in 5–4.

CHAPTER IX

FRICTION

78. Introduction. Friction may be defined as the phenomenon associated with the forces which are called into existence between two bodies when one of the bodies tends to slide or is caused to slide over the other. The force of friction acts along the tangent to the surfaces in contact. The direction of the force of friction acting on either of the bodies is such as to oppose the motion of that body over the other.

The subject of friction may be broadly classified under the following heads:

(1) Friction of fluids.
(2) Friction between solids completely separated by fluids.
(3) Friction between solids imperfectly separated by fluids.
(4) Friction between solids.

The friction of fluids is of importance in the study of the flow of liquids in conduits, in ship propulsion, and in other applications. A shaft completely or partially separated from its bearing by a film of oil affords examples of the second and third classes of friction respectively. The friction presented by the first three classes properly belongs to the subject of hydrodynamics. The friction between solids is essentially different from the friction of the first three classes.*

79. Laws of friction. The laws of friction between two bodies separated completely or partially by a fluid are diametrically opposed to the laws of friction between two solids. The friction between two bodies completely separated by a fluid is nearly independent of the load and the nature of the surfaces of the bodies, and it is dependent upon the relative velocity of the bodies and the physical characteristics of the fluid. When the rubbing surfaces are incompletely separated by a very thin film of lubricant, it has been found that certain oils are better

* T. E. Stanton's "Friction" (Longmans, Green & Co.) and H. Lamb's "Hydrodynamics" (Cambridge University Press).

lubricants when used with certain metals as bearing surfaces. If the surfaces of contact are very carefully cleaned and placed in air entirely free from moisture, the friction is very much greater than under ordinary circumstances. This fact has inclined modern investigators to replace the old theory that friction of clean surfaces is entirely due to the interlocking of the asperities of the surfaces in contact, by the theory that it is principally due to molecular attraction between the surfaces.

For clean or partially contaminated surfaces, the following laws hold with a fair degree of accuracy:

1. The frictional force F between two solid bodies in contact may be less than, but cannot exceed, a definite fraction μ of the normal pressure N between the bodies. The fraction μ is called the coefficient of static friction and is practically constant for given materials.

2. The frictional force F attains its greatest possible value, called the limiting friction, when the body is on the point of moving. The limiting friction is designated by F_l. The frictional force may vary between zero and F_l and is equal to the external force which tends to cause motion.

3. The limiting value of the frictional force F_l is (*a*) proportional to the normal pressure, (*b*) nearly independent of the area of contact, (*c*) dependent on the nature of the surfaces.

These three experimental results may be embodied in the formula

$$F_l = \mu N.$$

The formula $F_l = \mu N$ gives the value of the frictional force when and only when the body is on the point of moving.

4. The frictional force after motion begins is called *kinetic friction.* It is less than the limiting friction F_l and its magnitude diminishes somewhat as the velocity increases.

The formula $F = \mu N$ is valid for kinetic friction, where μ is now the coefficient of kinetic friction. The sudden diminution of friction when sliding takes place accounts for the fact that a train may be stopped in a shorter distance with the wheels braked and turning than is possible when the wheels are sliding on the rails.

80. Angle of friction; cone of friction. The resultant R of the limiting frictional force F_l and the normal pressure N is equal to $\sqrt{F_l^2 + N^2}$. The *angle of friction* is defined as the angle ϕ between the normal pressure N and the resultant R. If the

line of action of the force P changes its direction but remains in the horizontal plane of contact of the bodies, the locus of the lines of action of the resultant R is a cone whose semivertical angle is ϕ. This is called the *cone of friction*.

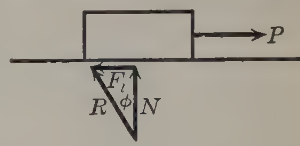

FIG. 174

Values of μ and ϕ follow:

MATERIALS	μ	ϕ
Wood upon wood	.2 to .5	11° to 27°
Wood upon metal	.2 to .6	11° to 31°
Metal upon metal	.15 to .25	8° to 14°

EXAMPLES

1. Determine the angle of inclination of a rough inclined plane so that a body on the plane is on the point of sliding down the plane.

Solution. The force which tends to move the body down the plane is $W \sin \theta$. The frictional force which prevents the body from sliding down the plane is $\mu W \cos \theta$, where μ is the coefficient of friction. When the body is on the point of moving,

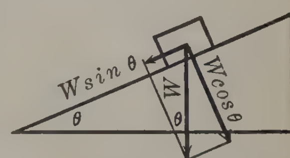

FIG. 175

$$W \sin \theta = \mu W \cos \theta.$$

Hence $$\tan \theta = \mu.$$

But, by definition, $\mu = \tan \phi$, where ϕ is the angle of friction. Hence if the plane is inclined at the angle of friction, the body will be on the point of moving.

2. A uniform rod of length $2\,l$ rests in a vertical plane with its lower end on a horizontal floor ($\mu = \mu_2$), and its upper end against a vertical wall ($\mu = \mu_1$), as shown in Fig. 176. Find the least angle θ which the rod can make with the floor before slipping begins.

Solution. Since the rod is about to slide, the frictional forces will be limiting frictional forces. If P and Q are the normal reactions at A and B respectively, the frictional forces are $\mu_1 P$ and $\mu_2 Q$.

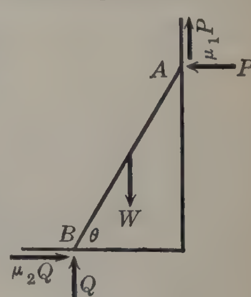

FIG. 176

$$\Sigma X = 0 \text{ gives } \mu_2 Q - P = 0.$$

$$\Sigma Y = 0 \text{ gives } Q + \mu_1 P - W = 0.$$

$$\Sigma M_A = 0 \text{ gives } Wl \cos \theta - 2\,Ql \cos \theta + 2\,\mu_2 Ql \sin \theta = 0.$$

Eliminating P and Q, $$\tan \theta = \frac{1 - \mu_1 \mu_2}{2\,\mu_2}.$$

3. Find the forces P necessary to move the wedges against the weight of 2000 lb. in Fig. 177, if the coefficient of friction between all surfaces is 0.3. The blocks C and D are held in position by smooth guides.

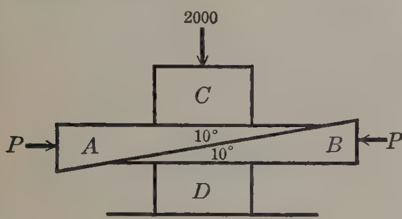

FIG. 177

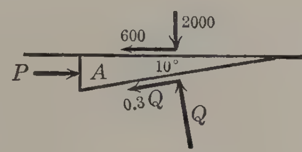

FIG. 178

Solution. The forces acting on the wedge A, when the 2000 lb. is being raised, are shown in Fig. 178.

$$\Sigma X = 0 \text{ gives } P - 600 - Q \sin 10° - \tfrac{3}{10} Q \cos 10° = 0.$$

$$\Sigma Y = 0 \text{ gives } Q \cos 10° - \tfrac{3}{10} Q \sin 10° - 2000 = 0.$$

From these equations, $P = 1606$ lb.

4. Find the force P acting at a distance R from the axis of a square-threaded screw to lift a load W if r is the mean radius of the screw, p is the pitch of the screw, and μ is the coefficient of friction.

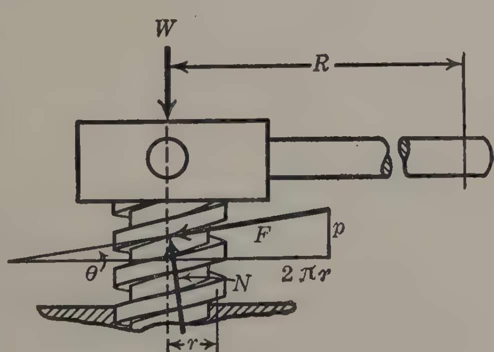

FIG. 179

Solution. Since the screw thread may be regarded as an inclined plane rolled upon a cylinder, the inclination θ of the plane is given by

$$\tan \theta = \frac{p}{2\pi r}.$$

Let the normal forces exerted on the screw by the nut be replaced by a single force N. The friction is represented by $F = \mu N$.

Resolving vertically, $N \cos \theta - F \sin \theta - W = 0$.

Taking moments about the axis of the screw,

$$PR = Fr \cos \theta + Nr \sin \theta.$$

Substituting from $F = \mu N$ and then eliminating N,

$$PR = \frac{Wr(\sin \theta + \mu \cos \theta)}{(\cos \theta - \mu \sin \theta)}.$$

PROBLEMS

1. The coefficient of friction between a particle resting on the surface of a fixed sphere and the sphere is $\frac{1}{\sqrt{3}}$. Show that the particle will remain at rest anywhere upon a spherical zone whose semivertical angle is 30° and whose apex is at the highest point of the sphere.

2. A uniform ladder 14 ft. long and weighing 48 lb. rests against a smooth vertical wall and a rough horizontal floor. ($\mu = 0.25$.) If the inclination of the ladder to the horizontal is arc tan 3, find (*a*) the least horizontal force which, applied at the lower end of the ladder, will cause the ladder to slide up the wall, and (*b*) the least horizontal force which will cause it to slide down the wall.

Ans. 20 lb., 4 lb.

3. A body weighing 100 lb. rests on a rough plane inclined at an angle of 30° with the horizontal, and the body is held in position by a rope making an angle of 30° with the inclined plane. What are the greatest and least tensions in the rope if the coefficient of friction is 0.25? *Ans.* 72.3 lb., 38.3 lb.

4. A uniform ladder weighing 50 lb. rests upon a horizontal floor at A and against a vertical wall at B. Find the least angle which the ladder can make with the floor so that a man weighing 150 lb. may ascend three fourths the length of the ladder without the ladder's slipping. ($\mu = 0.2$ for each surface.) *Ans.* $\theta =$ arc tan 3.38.

5. Two slender horizontal rods 2 ft. apart are fixed in a plane which makes an angle of 45° with the horizontal. A thin bar 10 ft. long lying in a plane perpendicular to the horizontal rods passes over the upper rod and under the lower rod. The bar is prevented from sliding down by friction. Find the least distance of the center of gravity of the bar from the upper rod consistent with equilibrium if $\mu = 0.3$. *Ans.* 2.33 ft.

6. A gate is supported by two rings which slip over a slender vertical rod. If the distance from the center of the rod to the center of gravity of the gate is 5 ft., find the distance between the rings when the gate will be on the point of slipping down the rod, if $\mu = 0.2$ at each ring. *Ans.* 2 ft.

7. A hollow cylinder weighing 60 lb. has an outer radius of 4 ft. and an inner radius of 2 ft. The cylinder rests with one end on a rough horizontal plane. ($\mu = 0.5$.) Find the least couple required to turn the cylinder around its axis. *Ans.* 93.3 lb.-ft.

8. A body weighing 100 lb. rests upon a plane inclined at 30° with the horizontal. ($\mu = \frac{1}{4}\sqrt{2}$.) The body is held in position by a horizontal force P. Find the greatest and least values of P, subject to the condition that the body does not move. *Ans.* 116.9 lb., 18.6 lb.

9. Three bodies, A, B, and C, weighing 10 lb., 20 lb., and 40 lb. respectively, rest upon a plane inclined at an angle θ with the horizontal. The bodies are connected by strings as shown in Fig. 180. If the coefficients of friction are 0.1, 0.2, and 0.5 respectively, determine θ for impending motion and also the tensions in the strings.
Ans. $\tan\theta = \frac{5}{14}$, $T_1 = 2.42$ lb., $T_2 = 5.38$ lb.

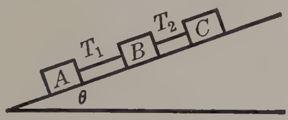

FIG. 180

10. A passenger coach weighing 60 T. and having two four-wheel trucks has a velocity of 60 mi. per hour. The coefficient of friction between a wheel and the rail is 0.25; the coefficient of friction between a brake-shoe and wheel is 0.11. If the total brake-shoe pressure is 90 per cent of the weight of the car, find the force on each wheel which tends to prevent rotation. Find the force which tends to prevent sliding on the rails if the wheels are locked.
Ans. 1485 lb., 3750 lb.

11. The weight of a vertical turbine shaft together with the parts carried by it is 200,000 lb. The diameter of the shaft is 12 in. The end of the shaft rests on a flat bearing. Find the moment necessary to rotate the shaft. ($\mu = 0.01$.) *Ans.* 667 lb.-ft.

12. Find the moment required to lower the load W of Example 4, page 115. *Ans.* $\dfrac{Wr(\sin\theta - \mu\cos\theta)}{\cos\theta + \mu\sin\theta}$.

13. A small block weighing 10 lb. rests upon a horizontal floor, the coefficient of friction being 0.4. Find the minimum force necessary to move the block, and the angle which it makes with the floor.
Ans. 3.71 lb., 21° 50′.

14. A body weighing 100 lb. rests on a plane inclined at arc sin 0.6 to the horizontal. The body is acted on by a horizontal force of 60 lb. Find the friction between the body and the plane. ($\mu = 0.8$.)
Ans. 12 lb.

15. A friction clutch consists of a frustum of a cone fitting inside a hollow cone. The coefficient of friction between the outer surface of the frustum and the inner surface of the hollow cone is 0.3, and the semivertical angle of the cones is 45°. The radii of the bases of the frustum are 4 in. and 8 in. respectively. The force which acts along the axis to keep the two members of the clutch engaged is 40 lb. Find the maximum moment of friction, assuming that the pressure is uniformly distributed over the surfaces.
Ans. 105.5 lb.-in.

16. Two cylinders, A weighing 100 lb. and B weighing 75 lb., rest upon planes inclined at angles θ_1 and θ_2 with the horizontal, as shown in Fig. 181. The cylinders are connected by a horizontal string in

the plane including the lines of greatest slope of the planes. If the cylinders are just on the point of slipping down and if the coefficient of friction between the cylinder A and its plane is 0.2, find the coefficient of friction between the cylinder B and its plane.

Ans. $\mu = \frac{4}{15}$.

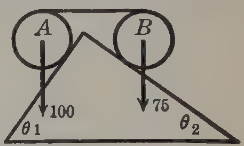

FIG. 181

17. Find the force P necessary to raise the weight W in the system of wedges shown in Fig. 182. The coefficient of friction for all rubbing surfaces is μ. Disregard the weights of A, B, and C.

Ans. $P = \dfrac{W(\sin 2\theta + \sin 4\phi)}{2 \sin 2\theta \cos 2\phi - \sin 4\phi}$, if $\tan \phi = \mu$.

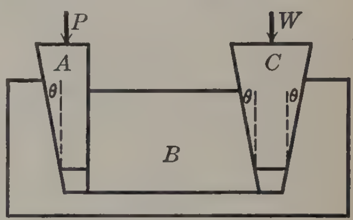

FIG. 182

18. A teeter board 18 ft. long, weighing 50 lb., rests horizontally with its middle point on the highest point of a rough horizontal cylinder of radius 3 ft. ($\mu = 0.2$.) The boy on one end weighs 30 lb. Between what limits must the weight of the boy on the other end lie in order that slipping may not occur? *Ans.* 23.2 lb. and 37.7 lb.

81. Belt friction. The problem of belt friction consists in determining the relation between the tensions in the two sides of a belt, the angle of contact of the belt, and the coefficient of friction.

Any infinitesimal element of a belt running at uniform speed will be in static equilibrium if its weight and the centrifugal force are neglected. The infinitesimal element AB of length $ds = r\,d\theta$ is in equilibrium under the tensions T and $T + dT$, the radial pressure of the pulley $p\,ds$, where p is the pressure per unit length, and the friction $\mu p\,ds$ acting along the arc ds. Resolving these forces along the radius which bisects the angle $d\theta$ gives

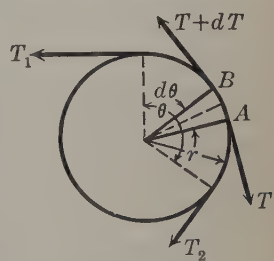

FIG. 183

$$(T + dT) \sin \frac{d\theta}{2} + T \sin \frac{d\theta}{2} = p\,ds. \qquad (1)$$

Taking moments about the center of the pulley gives

$$(T + dT)r - Tr = (\mu p\,ds)r. \qquad (2)$$

Replacing $\sin\frac{d\theta}{2}$ by $\frac{d\theta}{2}$, and neglecting infinitesimals of the second order, (1) becomes

$$T\,d\theta = p\,ds. \tag{3}$$

From (2),
$$dT = \mu p\,ds. \tag{4}$$

From (3) and (4),
$$\frac{dT}{T} = \mu\,d\theta. \tag{5}$$

This differential equation gives the relation between the change in the tension of the belt and the corresponding change in the angle of contact.

Integrating (5),

$$\log_e \frac{T_1}{T_2} = \mu(\theta_1 - \theta_2) = \mu\theta, \tag{6}$$

or
$$\frac{T_1}{T_2} = e^{\mu\theta}. \tag{7}$$

Equation (6), expressed in common logarithms, becomes

$$\log_{10} \frac{T_1}{T_2} = 0.4343\ \mu\theta. \tag{8}$$

Equations (6), (7), and (8) are valid only when slipping is impending or with the kinetic value of μ when the belt is slipping. The angle is always expressed in radians.

EXAMPLE

A rope passing over a fixed horizontal cylinder supports a 1000-pound weight. What vertical force is required to raise and lower the weight? ($\mu = 0.3$.)

Solution. Substituting in the equation $\log_{10} \frac{T_1}{T_2} = 0.4343\ \mu\theta$ gives

$$\log_{10} \frac{T_1}{T_2} = (0.4343)(0.3)(\pi) = 0.4094,$$

or
$$\frac{T_1}{T_2} = 2.57.$$

Hence
$$T_1 = (2.57)(1000) = 2570 \text{ lb.},$$

which is the force necessary to raise the 1000-pound weight.

The force required to lower the 1000-pound weight is

$$\frac{1000}{2.57} = 389 \text{ lb.}$$

PROBLEMS

1. A rope is wrapped halfway around a post. If the tension on one end of the rope is 200 lb., find the maximum and minimum tensions on the other end when motion impends. ($\mu = 0.2$.) *Ans.* 375 lb., 106.8 lb.

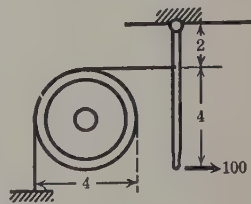

FIG. 184

2. A force of 10 lb. at one end of a rope is just sufficient to support a weight of 20 lb. on the other end when the rope contacts an arc of 120° of a fixed horizontal cylinder and the coefficient of friction is μ. If the coefficient of friction and the angle of contact were each doubled, how much would the 10-pound force support? *Ans.* 160 lb.

3. A belt which passes over a fixed pulley is just on the point of slipping when the tensions are 200 lb. and 50 lb. If the angle of contact is 200°, find the coefficient of friction. *Ans.* $\mu = 0.4$.

4. The angle of contact of the brake band of the brake shown in Fig. 184 is 90° and $\mu = 0.3$. If a force of 100 lb. is applied at the handle, find the frictional moment developed by the brake if the wheel turns (*a*) clockwise, (*b*) counterclockwise.

Ans. 361 lb.-ft., 226 lb.-ft.

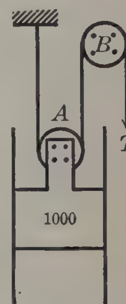

FIG. 185

5. In Fig. 185 the 1000-pound weight is constrained to move vertically between smooth guides. The coefficient of friction between the rope and the nonrotating drums A and B is 0.3. Find the tensions required to raise and to lower the weight. *Ans.* 1843 lb., 109.6 lb.

6. Three rough pegs are fastened to a block B weighing 500 lb., which slides vertically in smooth guides A. A cord passes over the pegs as shown in Fig. 186. Find the weight P which must be attached to the cord so that the block B may be just on the point of moving downward. ($\mu = 0.4$.)

Ans. 199 lb.

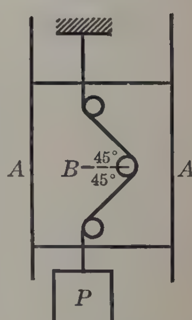

FIG. 186

7. A rope is wound around a rough post through any given angle θ. If the tension T on one end of the rope is constant, prove that $T = \sqrt{T_1 T_2}$, where T_1 and T_2 are the maximum and minimum values of the tension on the other end of the rope when slipping impends.

8. A rope is wrapped three times around a post. The tension on one end of the rope is 100 lb. and on the other 200 lb. The coefficient of friction is 0.5. Find the total frictional force acting on the rope.

9. A man weighing 160 lb. stands in a basket weighing 40 lb. attached to one end of a rope. The rope passes over a rough horizontal cylinder ($\mu = 0.3$) and vertically downward into the hands of the man, who slowly lowers himself. Find the pull which the man exerts on the rope. *Ans.* 56.1 lb.

82. Rolling friction. Osborne Reynolds, in his "Scientific Papers," Vol. I, has shown that when the weight borne by a wheel or roller resting upon a plane is not great enough to produce a permanent deformation in either the roller or the plane, the resistance to rolling arises as follows: Since no material is perfectly hard, the roller and the plane surface will each undergo a change of form as shown, greatly exaggerated, by the full line CD in Fig. 187. This deformation causes the material underneath the roller to slip slightly with reference to the roller while in contact with it. The friction produced by this slippage causes the resistance to rolling.

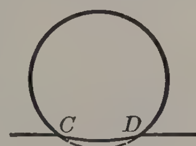

FIG. 187

83. Measure of rolling resistance. The usual method of measuring rolling resistance, and one which does not take into account Osborne Reynolds's theory, may be described as follows:

Let the resultant R of the forces between the roller and the surface act at A, Fig. 188. Taking moments about A gives

$$P(OC) = Wa.$$

Since OC is nearly equal to the radius r of the roller,

$$P = \frac{Wa}{r}.$$

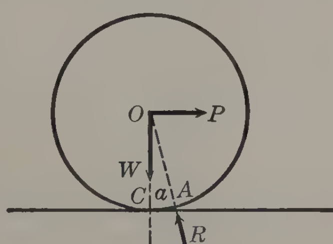

FIG. 188

The values of a are usually given in inches. Since for the same materials the values of a vary considerably for different loads and radii, they should be used with caution. The distance a is called the coefficient of rolling friction. It is not, however, a coefficient of friction as usually defined.

VALUES OF a. (GOODMAN)

Iron or steel wheels on iron or steel rails	.007 in. to .015 in.
Iron or steel wheels on wood	.060 in. to .100 in.
Pneumatic tires fully inflated on good roads	.020 in. to .022 in.

PROBLEMS

1. If the coefficient of rolling friction is 0.01 in., what horizontal force must be applied at the center of a wheel 30 in. in diameter and weighing 500 lb. to just cause it to roll? *Ans.* 0.33 lb.

2. The body, gear, and load of a dump car weigh 2 T. The car has four wheels each 30 in. in diameter. Each pair of wheels with their axle weighs 600 lb. The diameter of each axle is 5 in. The coefficient of friction between an axle and its bearing is 0.06, and the coefficient of rolling friction is 0.02 in. Find the horizontal force required to just cause motion of the car. *Ans.* 46.8 lb.

3. A horizontal slab weighing 5000 lb. rests upon two rollers 10 in. in diameter and weighing 200 lb. each. The rollers are supported by a horizontal track. If the coefficient of rolling friction between the slab and the rollers and between the rollers and the track is 0.05 in. and 0.06 in. respectively, find the horizontal force which must be applied to the slab to just cause motion. *Ans.* 57.4 lb.

4. A car weighing 70,000 lb. is carried on eight journals each 5 in. in diameter and 7 in. long. Find the drawbar pull required to start the car if the coefficient of friction is 0.12 for starting and the wheels are 30 in. in diameter. The coefficient of rolling friction is 0.02 in. *Ans.* 1493 lb.

CHAPTER X

THE CATENARY

84. Definition. The curve which a heavy flexible string or cable assumes when suspended from two points is called a *common catenary.* In the design of electric transmission lines and suspension bridges it is necessary to calculate the tension in a loaded cable. The load may consist of the weight of the cable itself or of an ice load, a wind load, or a roadway suspended from the cable. The load is assumed to be uniformly distributed *along the cable* and the cable is assumed to be flexible.

85. The equation of the catenary. Let ABC represent the catenary, where B is the lowest point or the point where the tangent to the curve is horizontal. Take the vertical through B as the y axis and some horizontal line Ox as the x axis. The distance OB will be determined later, so that Ox will represent the directrix of the curve. Let the tangent at any point $P(x, y)$ of the curve make an angle ψ with the positive x axis, so that

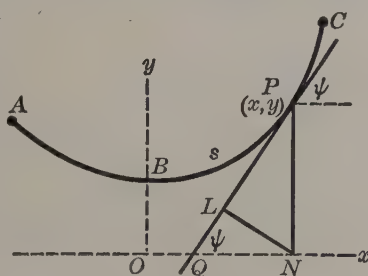

FIG. 189

$\tan \psi = \dfrac{dy}{dx}$. Let T_0 and T be the tensions in the cable at the points B and P respectively. Also, let s be the length of the arc BP, and let w be the weight of a *unit length of the cable,* so that ws is the weight of the arc BP.

The forces acting upon the arc BP, considered as a free body, are the tensions T_0 and T, acting along the tangents to the arc at B and P respectively, and the weight ws, acting at the center of gravity of the arc. The equations of equilibrium give

$$T \sin \psi = ws, \tag{1}$$

$$T \cos \psi = T_0. \tag{2}$$

Hence
$$\tan \psi = \frac{ws}{T_0}. \tag{3}$$

Introducing a new constant c defined by $T_0 = wc$, (3) becomes

$$s = c \tan \psi = c \frac{dy}{dx}, \tag{4}$$

which is known as the intrinsic equation of the curve.

From the differential relation $(ds)^2 = (dy)^2 + (dx)^2$,

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{c^2}{s^2}}$$

from (4). Hence $$dy = \pm \frac{s\,ds}{\sqrt{s^2 + c^2}},$$

and, by integration, $y + K = \sqrt{s^2 + c^2}$.

The positive sign of the radical is chosen since y increases with s.

The origin is now chosen so that the constant of integration K is zero. This makes $y = c$ when $s = 0$.

Hence, finally, $$y^2 = s^2 + c^2. \tag{5}$$

In a similar manner, proceeding from the differential relation above,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{s^2}{c^2}}.$$

Hence $$dx = \frac{c\,ds}{\sqrt{s^2 + c^2}},$$

and, by integration,

$$x + K' = c \log\left[s + \sqrt{s^2 + c^2}\right].$$

Since $s = 0$ when $x = 0$,

$$K' = c \log c.$$

Therefore $$\frac{x}{c} = \log\left[\frac{s + \sqrt{s^2 + c^2}}{c}\right]. \tag{6}$$

In exponential form, (6) becomes

$$ce^{\frac{x}{c}} = s + \sqrt{s^2 + c^2}. \tag{7}$$

Eliminating s between (5) and (7) gives

$$ce^{\frac{x}{c}} - y = \sqrt{y^2 - c^2}.$$

Squaring and simplifying,

$$c^2 e^{\frac{2x}{c}} - 2\,yce^{\frac{x}{c}} = -\,c^2.$$

Solving for y, $$y = \frac{c}{2}\left[e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right], \tag{8}$$

which is the equation of the common catenary.

From (5), $$s = \sqrt{y^2 - c^2}.$$

Substituting the value of y from (8),

$$s = \sqrt{\frac{c^2}{4}\left[e^{\frac{2x}{c}} + 2 + e^{-\frac{2x}{c}}\right] - c^2}.$$

Hence $$s = \frac{c}{2}\left[e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right]. \tag{9}$$

Equations (8) and (9) are conveniently expressed in hyperbolic functions as

$$y = c \cosh \frac{x}{c}, \tag{10}$$

$$s = c \sinh \frac{x}{c}. \tag{11}$$

86. Properties of the catenary. *Mechanical properties.* Squaring (1) and (2), § 85, and adding,

$$T = \sqrt{w^2 s^2 + T_0{}^2}.$$

Replacing T_0 by wc, as before,

$$T = w\sqrt{s^2 + c^2} = wy. \tag{1}$$

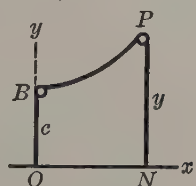

FIG. 190

The equation $T = wy$ shows that the tension at any point P is equal to the weight of the cable of length y. In other words, if the cable is allowed to run over a smooth peg at any point P and hang vertically downward and is then cut off at the x axis, the weight of the vertically hanging portion will maintain the catenary between P and B as before. The length necessary to maintain the tension at the lowest point B is accordingly equal to c.

Equation (2), § 85, $T \cos \psi = T_0 = wc$, shows that the *horizontal* component of the tension at any point is constant, namely, T_0, or the weight of the cable of length c.

Equation (1), § 85, $T \sin \psi = ws$, shows that the *vertical* component of the tension at any point P is the weight of the cable of the length of the arc from P to B.

Hence at any point P,

$$\left.\begin{array}{l}\textit{Total}\text{ tension} = \text{weight of cable of length } y,\\ \text{Vertical component of tension} = \text{weight of cable of length } s,\\ \text{Horizontal component of tension} = \text{weight of cable of length } c.\end{array}\right\} \tag{2}$$

Geometric properties. From N, the foot of the ordinate of any point P, drop a perpendicular NL to the tangent at P; then the angle $PNL = \psi$.

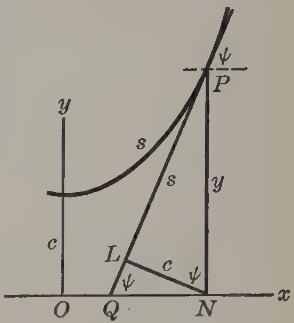

FIG. 191

Substituting the value $T = wy$ from (1) in (1) and (2), § 85, and replacing T_0 by wc,

$$\left.\begin{aligned} y \sin \psi &= s \text{ and } y \cos \psi = c. \\ \text{Hence} \qquad PL &= s \text{ and } NL = c. \end{aligned}\right\} \quad (3)$$

The relations between s, y, c, and ψ may easily be memorized by the aid of the right triangle PLN.

87. To find the equation of the catenary and its total length, having given the inclination of the catenary at the points of suspension and the span, the points of suspension being on the same level. In Fig. 192 let

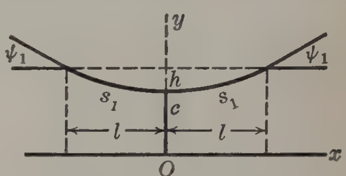

FIG. 192

$\tan \psi_1 = p$,

$2\,s_1 =$ total length of cable,

$2\,l =$ total span,

$h =$ the drop or sag.

From (10) or (8), § 85,

$$y = c \cosh \frac{x}{c}.$$

Therefore $$\frac{dy}{dx} = \sinh \frac{x}{c} \quad \text{or} \quad \tfrac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$$

Hence $$e^{\frac{l}{c}} - e^{-\frac{l}{c}} = 2\,p.$$

Multiplying by $e^{\frac{l}{c}}$, $$e^{\frac{2l}{c}} - 2\,pe^{\frac{l}{c}} - 1 = 0.$$

Hence $$e^{\frac{l}{c}} = p + \sqrt{p^2 + 1},$$

the positive sign being chosen since $e^{\frac{l}{c}}$ cannot be negative.

Hence $$\frac{l}{c} = \log_e \left(p + \sqrt{p^2 + 1}\right),$$

and $$c = \frac{l}{\log_e \left(p + \sqrt{p^2 + 1}\right)}. \qquad (1)$$

Equation (1) gives the parameter c of the catenary in terms of the given quantities p and l. Since c is obtained, the equation of the catenary,

$$y = c \cosh \frac{x}{c},$$

is therefore known. From Fig. 192 the coördinates of the right-hand support are $x = l$ and $y = h + c$. Substituting these values in the equation of the catenary gives

$$h + c = c \cosh \frac{l}{c}. \tag{2}$$

The ordinate $h + c$ is therefore known and serves to locate the origin. Also, since c is known, the sag h is known.

Substituting the values s_1 and ψ_1 in (4), § 85,

$$s = c \tan \psi,$$

gives $$2\, s_1 = 2\, c \tan \psi_1 = 2\, cp = \frac{2\, lp}{\log_e (p + \sqrt{p^2 + 1})}.$$

Hence the length of cable is given by

$$2\, s_1 = \frac{2\, lp}{\log_e (p + \sqrt{p^2 + 1})}. \tag{3}$$

88. To find the equation of the catenary, having given its length and span, the points of suspension being on the same level. The equation of the catenary is

$$y = c \cosh \frac{x}{c}.$$

The parameter c is to be determined in terms of the span $2\, l$ and the length of cable $2\, s_1$. At the right-hand support $x = l$ and $s = s_1$. Substituting these values in (11), § 85, gives the equation

$$s_1 = c \sinh \frac{l}{c}$$

for the determination of c.

This equation can be solved by trial, making use of a table of natural hyperbolic functions.

For a first approximation, assume $c = 1$, 10, 100, 1000, 10,000 in succession and calculate the corresponding values of $c \sinh \frac{l}{c}$. Usually some of these values will be larger and others will be smaller than the given s_1, which will show that c lies between two of the successively assumed values, say between 100 and 1000.

Next assume $c = 200$, 300, etc. and proceed as before, continuing until a sufficiently accurate value of c is determined.

The equation can also be solved by graphical methods. Transforming the equation

$$s_1 = c \sinh \frac{l}{c}$$

by letting $c = \frac{1}{x}$, it becomes

$$s_1 x = \sinh lx.$$

Plot, in rectangular coördinates, the two curves $y = s_1 x$ and $y = \sinh lx$, as shown in Fig. 193.

The abscissa of the point of intersection is the value of x which satisfies the transformed equation, and c is the reciprocal of x.

It will be found that when s_1 and l are nearly equal, c will be large, and when s_1 is much larger than l, c will be small.

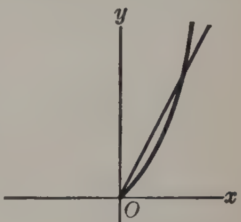

FIG. 193

EXAMPLE

Given $s_1 = 50$ ft. and $l = 40$ ft., to determine c.

Solution. Making the substitution in the equation

$$s_1 = c \sinh \frac{l}{c}$$

gives

$$50 = c \sinh \frac{40}{c},$$

from which c is approximately determined by repeated trials as follows:

$10 \sinh 4.00 = 272.9$	$c = 10$
$100 \sinh 0.40 = 41.08$	$c = 100$
$1000 \sinh 0.04 = 40.0$	$c = 1000$
$50 \sinh 0.80 = 44.4$	$c = 50$
$40 \sinh 1.00 = 47.0$	$c = 40$
$38 \sinh 1.05 = 47.65$	$c = 38$
$35 \sinh 1.14 = 49.11$	$c = 35$
$33 \sinh 1.21 = 50.4$	$c = 33$
$34 \sinh 1.18 = 50.1$	$c = 34$

89. General case. To find the equation of the catenary when the span, the length, and the difference of elevation of the two supports are given.

Let D be the lowest point and A and B the points of suspension of the catenary shown in Fig. 194. Let the coördinates of A referred to O be (x_1, y_1); also let s_1 be the length. Let the

coördinates of B referred to O be (x_2, y_2), and let s_2 be the length. Contrary to the usual custom, let both x_1 and x_2 be positive.

Then $x_1 + x_2 = S$, the total span. (1)

Let $x_1 - x_2 = R$, an unknown quantity. (2)

By addition and subtraction,

$$x_1 = \frac{S+R}{2}, \qquad (3)$$

$$x_2 = \frac{S-R}{2}. \qquad (4)$$

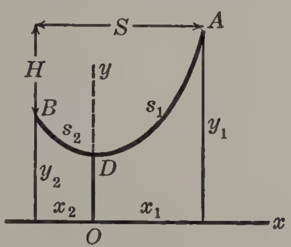

FIG. 194

Also let $y_1 - y_2 = H$, the difference of elevation of supports, (5)

and $s_1 + s_2 = L$, the total length of cable. (6)

From the mechanical properties of the catenary it is evident that the right and left portions of this catenary are one and the same catenary; in other words, both portions have the same parameter c.

Substitution of the particular values in the equation

$$s = c \sinh \frac{x}{c}$$

gives

$$s_1 = c \sinh \frac{S+R}{2c},$$

$$s_2 = c \sinh \frac{S-R}{2c}.$$

By addition,

$$L = s_1 + s_2 = c\left[\sinh \frac{S+R}{2c} + \sinh \frac{S-R}{2c}\right]. \qquad (7)$$

Substituting in a similar manner in the equation

$$y = c \cosh \frac{x}{c}$$

gives

$$y_1 = c \cosh \frac{S+R}{2c},$$

$$y_2 = c \cosh \frac{S-R}{2c}.$$

By subtraction,

$$H = y_1 - y_2 = c\left[\cosh \frac{S+R}{2c} - \cosh \frac{S-R}{2c}\right]. \qquad (8)$$

Transforming (7) and (8) gives

$$L = 2\,c \sinh \frac{S}{2\,c} \cosh \frac{R}{2\,c}, \tag{9}$$

$$H = 2\,c \sinh \frac{S}{2\,c} \sinh \frac{R}{2\,c}. \tag{10}$$

Squaring (9) and (10) and subtracting gives

$$L^2 - H^2 = 4\,c^2 \sinh^2 \frac{S}{2\,c},$$

since $\cosh^2 a - \sinh^2 a = 1$.

Hence
$$\sqrt{L^2 - H^2} = 2\,c \sinh \frac{S}{2\,c}. \tag{11}$$

In this equation L, H, and S are known; therefore $2\,c$ can be found by trial, as in the previous case. Dividing (10) by (9),

$$\frac{H}{L} = \tanh \frac{R}{2\,c}, \tag{12}$$

from which R may be found by the use of a table of hyperbolic functions.

From (3) and (4), x_1 and x_2 may be found. On substituting the values of x_1 and c in the equation

$$y = c \cosh \frac{x}{c},$$

y_1 is determined, which locates the origin and lowest point. The tensions at various points can then be found as usual.

90. Parabolic cable; load distributed uniformly over the span. A cable that is uniformly loaded along the horizontal with a load that is large in comparison with the weight of the cable will assume approximately the form of a parabola. (Suspension bridges approximate this system of loading.)

Let O be the lowest point of the cable, $P(x, y)$ any other point, and w the load per unit *horizontal distance*, as shown in Fig. 195.

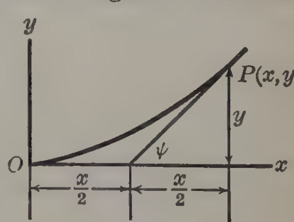

FIG. 195

Let T_0 and T be the tensions in the cable at the points O and P respectively. The forces acting on the arc OP are the

tensions T_0 and T acting along the tangents to the arc at O and P respectively, and the weight wx acting at a point whose abscissa is $\frac{x}{2}$.

The equations of equilibrium for the arc are

$$T \cos \psi = T_0, \tag{1}$$

$$T \sin \psi = wx. \tag{2}$$

Hence
$$\tan \psi = \frac{dy}{dx} = \frac{w}{T_0} x.$$

Integrating,
$$y = \frac{wx^2}{2\,T_0} + K.$$

Since $x = 0$ where $y = 0$, therefore $K = 0$. Hence the equation of the curve is

$$y = \frac{wx^2}{2\,T_0}. \tag{3}$$

Squaring (1) and (2) and adding gives

$$T = \sqrt{T_0{}^2 + w^2x^2}, \tag{4}$$

which expresses the tension at any point (x, y) in terms of the tension at the lowest point.

91. Approximation to the catenary. Expanding $\cosh \frac{x}{c}$ in a series,

$$y = c \cosh \frac{x}{c} = c\left(1 + \frac{x^2}{2\,c^2} + \frac{x^4}{24\,c^4} + \cdots\right).$$

When the sag is small compared with the span, c is large in comparison with x. Hence $\frac{x}{c}$ is a small quantity, and therefore an approximation to the catenary may be obtained by taking only the first two terms in the series, which gives

$$y = c + \frac{x^2}{2\,c}.$$

By moving the origin c units upward, the equation becomes

$$y = \frac{x^2}{2\,c}.$$

Replacing c by its value $\frac{T_0}{w}$, the equation becomes

$$y = \frac{wx^2}{2\,T_0},$$

which is the equation of the parabola in § 90.

PROBLEMS

1. A balloon is fastened to the ground by a rope 500 ft. in length. The rope weighs 2 lb. per foot, and the angle of inclination at the ground is 60°. The tension in the rope at the ground is 200 lb. Find the vertical height of the balloon above the level ground.

Ans. 489 ft.

2. A guy wire has one end fastened to the ground and the other end attached to a tower 120 ft. high. The wire weighs 0.2 lb. per foot, and the tension in the wire at the ground is 500 lb. Find the length of the guy wire if the angle which it makes with the horizontal at the lower end is arc cos 0.4. *Ans.* 130 ft.

3. One car is towed by another at a constant speed. The rope is attached to each car at a height of 2 ft. above the ground, and the middle of the rope is 1 ft. above the ground. Find the horizontal tension in the rope if the rope is 20 ft. long and weighs 1 lb. per foot.

Ans. 49.5 lb.

4. A cable weighing 0.5 lb. per foot is suspended between two points on the same level and 100 ft. apart. If the angle of inclination of the cable is arc tan 0.75 with the horizontal at the supports, find the equation of the catenary and its total length.

Ans. $y = 72.1 \cosh \frac{x}{72.1}$, 108.2 ft.

5. A cable is suspended between two points on the same level. The inclination of the cable at one end is 60° with the horizontal, and the horizontal distance to the nearest point where the inclination is 45° is 50 ft. Find the span. *Ans.* 302.3 ft.

6. A cable 140 ft. long is suspended from two points at the same level and 100 ft. apart. Find the equation of the catenary and the sag at the middle. *Ans.* $y = 34 \cosh \frac{x}{34}$, 43.9 ft.

7. A cable 150 ft. long is suspended between two points, A and B. If B is selected as the origin of rectangular axes, the coördinates of A relative to B are $x = 100$ ft., $y = 40$ ft. Find the equation of the catenary referred to the usual axes. *Ans.* $y = 32.5 \cosh \frac{x}{32.5}$.

8. A cable weighing 1 lb. per foot is suspended between two points on the same level 100 ft. apart. The sag at the middle is 20 ft. Find the length of the cable and the tension at the points of support.

Ans. $c = 65.59$, 109.97 ft., 85.59 lb.

9. The span of a suspension bridge is 1000 ft., and the sag at the middle is 100 ft. The load along the horizontal borne by each cable is 800 lb. per foot. Find the tension in each cable at the points of support and at its lowest point. *Ans.* 1,077,033 lb., 1,000,000 lb.

CHAPTER XI

WORK

92. Definition. An important distinction exists between the usual concept of work in daily life and the definition of work in mechanics. A man is said to do work if he carries a weight while walking on a level floor, although no work is done on the weight in the sense in which the word "work" is used in mechanics. No work is done on a body by those forces which are perpendicular to the direction of motion of the body. No work is done on a body by any force if the body remains at rest.

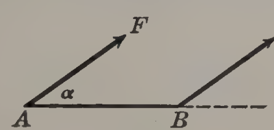

FIG. 196

The work of a constant force is defined as the product of the displacement of its point of application and the component of the force in the direction of the displacement. Thus, let a constant force F make a constant angle α with the direction of the displacement of its point of application, AB, and let the straight line AB be the displacement; then the work done by the force F is expressed by the equation

$$W = F \cos \alpha \cdot AB. \tag{1}$$

The point of application A is usually a point of some body which moves along the path AB. Other forces may act at A or at other points of the body without altering the work done by the force F.

The following special cases and modifications of equation (1) lead to a more general case:

1. If the angle α is zero and the force F is constant, the work done is

$$W = F \cdot AB.$$

If the angle α is zero and the force F is variable, then the work done is

$$W = \int F\,dx,$$

where dx is an element of the straight-line path AB. In order to effect the integration, the law of variation of F with x must be determined; that is, F must be expressed as a function of x.

2. If the angle α is 90° and the point of application of the force F moves along AB, the work done by the force F is zero.

3. If the angle α is greater than 90°, the work done by the force F while its point of application moves along AB is negative. Negative work done by a force on a body is interpreted as work done by the body against the force.

4. The equation (1), $W = F \cos \alpha \cdot AB,$

may be written $W = F \cdot AB \cos \alpha.$

Work may therefore be defined as the product of the force and the projection of the displacement on the line of action of the force.

General case. When the force F is variable and the angle α is variable and the path of displacement is a curve, the work done by the force F while the point of application traverses the curve from a point s_0 to a point s_1 is expressed by the line integral

$$W = \int_{s_0}^{s_1} F \cos \alpha \cdot ds.$$

EXAMPLES

1. The modulus of a spring is 10 lb. per inch. Find the work done in stretching the spring 4 in. if the initial stretch in the spring is 2 in.

Solution. The force that stretches the spring is variable. It requires 10 lb. to keep the spring stretched 1 in. longer than its natural length, 20 lb. to keep it stretched 2 in. longer, and so on. As long as the force and elongation obey Hooke's law, the force is proportional to the elongation. Hence

$$F = 10\,x,$$

where x measures the amount of elongation. The work done in stretching the spring from an initial elongation of 2 in. to a final elongation of 6 in. is therefore

$$\int_2^6 10\,x\,dx = \Big[5\,x^2\Big]_2^6 = 180 - 20 = 160 \text{ in.-lb.}$$

When the force is a linear function of the displacement, the work done may be found by multiplying the displacement by the average of the forces at the beginning and end of the displacement. In this case the initial and final forces are 20 lb. and 60 lb. respectively and the average force is 40 lb. Hence the work stored in the spring during displacement is $40 \times 4 = 160$ in.-lb.

This method of replacing the process of integration by the product of the displacement and the average of the initial and final forces is applicable only when the force is a *linear* function of the displacement.

2. A box weighing w lb. is dragged at constant speed along a floor a distance d ft. by means of a rope. The rope makes a constant angle α with the floor, and the coefficient of sliding friction is μ. Find the work done.

Solution. The static conditions are

$$P \cos \alpha = \mu N,$$

$$N + P \sin \alpha = w,$$

from which

$$P = \frac{\mu w}{\mu \sin \alpha + \cos \alpha}.$$

Hence the work done is

$$W = Pd \cos \alpha = \frac{\mu w d \cos \alpha}{\mu \sin \alpha + \cos \alpha}.$$

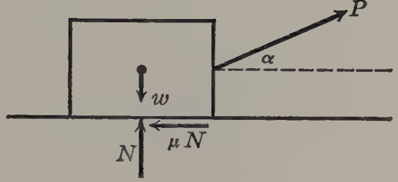

FIG. 197

3. In Example 2 let the rope be drawn over a small pulley which is fixed in position, whereby the angle α becomes variable. Find the work done.

Solution. The work done is given by

$$W = \int_{\alpha_1}^{\alpha_2} P \cos \alpha \, ds.$$

From Example (2),

$$P = \frac{\mu w}{\mu \sin \alpha + \cos \alpha}.$$

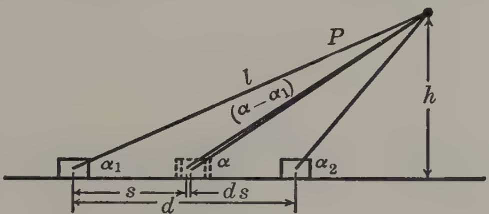

FIG. 198

In order to effect the integration, ds must be expressed in terms of the variable α. From the triangle whose base is the variable s, the Sine Law gives

$$\frac{s}{\sin(\alpha - \alpha_1)} = \frac{l}{\sin \alpha}.$$

Hence $$s = \frac{l \sin(\alpha - \alpha_1)}{\sin \alpha} = l \cos \alpha_1 - l \sin \alpha_1 \cot \alpha$$

and, by differentiation, $$ds = l \sin \alpha_1 \csc^2 \alpha \, d\alpha.$$

Hence
$$\begin{aligned}
W &= \int_{\alpha_1}^{\alpha_2} \left(\frac{\mu w}{\mu \sin \alpha + \cos \alpha} \right) (\cos \alpha)(l \sin \alpha_1 \csc^2 \alpha \, d\alpha) \\
&= \mu w l \sin \alpha_1 \int_{\alpha_1}^{\alpha_2} \frac{\cos \alpha \csc^2 \alpha \, d\alpha}{\mu \sin \alpha + \cos \alpha} \\
&= \mu w l \sin \alpha_1 \int_{\alpha_1}^{\alpha_2} - \frac{\cot \alpha \, d(\cot \alpha)}{\mu + \cot \alpha} \\
&= \mu w l \sin \alpha_1 \int_{\alpha_1}^{\alpha_2} \left(-1 + \frac{\mu}{\mu + \cot \alpha} \right) d(\cot \alpha) \\
&= \mu w l \sin \alpha_1 \Big[-\cot \alpha + \mu \log(\mu + \cot \alpha) \Big]_{\alpha_1}^{\alpha_2} \\
&= \mu w l \sin \alpha_1 \left[\cot \alpha_1 - \cot \alpha_2 + \mu \log \frac{\mu + \cot \alpha_2}{\mu + \cot \alpha_1} \right].
\end{aligned}$$

Replacing $l \sin \alpha_1$ by h,

$$W = \mu w h \left[\cot \alpha_1 - \cot \alpha_2 + \mu \log \frac{\mu + \cot \alpha_2}{\mu + \cot \alpha_1} \right].$$

4. Find the work done per stroke on the crank pin B by the steam pressure in the engine shown in Fig. 199. Steam is admitted throughout the stroke at a constant pressure of 100 lb. gauge. Inertia and friction forces are neglected.

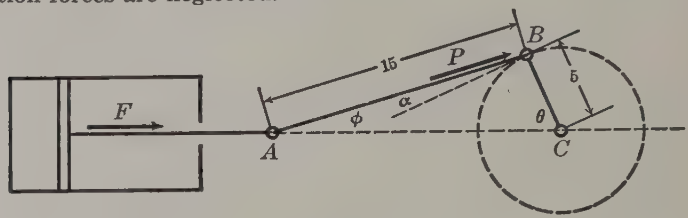

FIG. 199

Solution. The work done is, by definition,

$$W = \int_{s=0}^{s=5\pi} P \cos \alpha \, ds, \tag{1}$$

where P is the axial stress in the connecting rod. In order to effect the integration, the variables P, $\cos \alpha$, and ds must each be expressed in terms of a common variable. By means of the geometry of the figure each of these may be expressed in terms of the crank angle θ as follows:

From the Sine Law, $$\frac{\sin \phi}{5} = \frac{\sin \theta}{15}.$$

Hence $$\sin \phi = \tfrac{1}{3} \sin \theta, \tag{2}$$

$$\cos \phi = \tfrac{1}{3} \sqrt{9 - \sin^2 \theta}, \tag{3}$$

$$\sec \phi = \frac{3}{\sqrt{9 - \sin^2 \theta}}. \tag{4}$$

Also $$\phi + \theta + \alpha = 90°,$$
and hence

$$\cos \alpha = \cos [90 - (\theta + \phi)] = \sin (\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi,$$

or $$\cos \alpha = \frac{\sin \theta}{3} (\cos \theta + \sqrt{9 - \sin^2 \theta}). \tag{5}$$

Since the crank pin B travels on a circle,

$$ds = r \, d\theta = 5 \, d\theta, \tag{6}$$

the limits of θ being 0 and π for one stroke.

From the equilibrium of the forces acting on the crosshead pin A,

$$F = P \cos \phi,$$

or, from (4), $$P = F \sec \phi = F \frac{3}{\sqrt{9 - \sin^2 \theta}}.$$

But the total pressure on the piston is

$$F = 100 (16 \pi) = 1600 \pi \text{ lb.}$$

Hence $$P = \frac{4800 \pi}{\sqrt{9 - \sin^2 \theta}}. \tag{7}$$

Substituting the results of (5), (6), and (7) in (1) gives

$$W = \int_0^{\pi} \left(\frac{4800\,\pi}{\sqrt{9-\sin^2\theta}}\right)\left[\frac{\sin\theta}{3}(\cos\theta + \sqrt{9-\sin^2\theta})\right](5\,d\theta)$$

$$= 8000\,\pi \int_0^{\pi}\left(\frac{\sin\theta\cos\theta}{\sqrt{9-\sin^2\theta}} + \sin\theta\right)d\theta$$

$$= 8000\,\pi\Big[-\sqrt{9-\sin^2\theta} - \cos\theta\Big]_0^{\pi}$$

$$= 16{,}000\,\pi = 50{,}265 \text{ in.-lb.}$$

Check. Since by the assumptions no energy is lost in friction, the total work done on the crank pin should be equal to the total work done on the piston by the steam. The latter is the total pressure F multiplied by the length of stroke, 10 in., which gives

$$1600\,\pi \times 10 = 16{,}000\,\pi = 50{,}265 \text{ in.-lb.}$$

5. Find the work done by a gas expanding in a cylinder against a piston from an initial volume v_1 to a final volume v_2, assuming that the gas expands according to the law $pv = k$.

Solution. If a is the area of the piston and p the pressure per unit area, the total pressure on the piston is pa. The work done on the piston while it moves through a distance ds is $pa\,ds$. But $a\,ds = dv$, where dv is the change in volume of the gas. Hence the work done is

$$W = \int_{v_1}^{v_2} p\,dv.$$

From the law $pv = k$, $\qquad p = \frac{k}{v}.$

Therefore the work is

$$W = \int_{v_1}^{v_2}\frac{k\,dv}{v} = \Big[k\log_e v\Big]_{v_1}^{v_2} = k\log_e\frac{v_2}{v_1},$$

and since $\qquad p_1v_1 = p_2v_2 = k,$

the work done is $\quad W = p_1v_1\,\log_e\frac{v_2}{v_1} = 2.303\,p_1v_1\log_{10}\frac{v_2}{v_1}.$

93. Graphical representation of work. If any two quantities are laid off to scale on two perpendicular lines, the area of the rectangle whose sides are these quantities will represent the product of the two quantities. Since work is defined as the product of a *displacement* and the *component of a force* in the direction of the displacement, it may be represented by an area.

If the force is constant or if its component in the direction of the displacement is constant, the work done is represented by the area of a rectangle.

If the force is proportional to the displacement, as in the case of the compression of a spring, the work done is represented by the area of a triangle or by the area of a trapezoid, according as the initial force is zero or not.

In general, the force may be any function of the displacement. When the force is plotted as a function of the displacement it

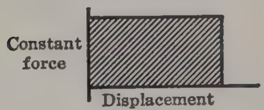

FIG. 200

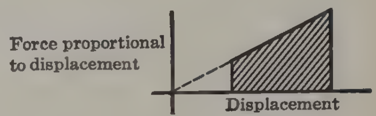

FIG. 201

is evident that the work done is represented by the area under the curve. In general, the area is obtained by integration. Only when the graphical representation of the force as a function of the displacement is a *straight line* may the process of integration be replaced by elementary geometric methods. The areas in Figs. 200 and 201 are obtained by multiplication of the

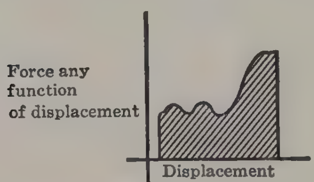

FIG. 202

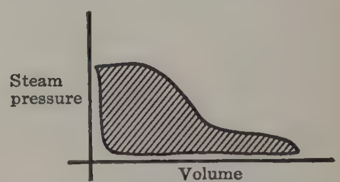

FIG. 203

displacement by the average of the initial and final forces. The area in Fig. 202 can be obtained by integration only.

The work of a steam engine is determined by an indicator, which is an instrument for continuously and simultaneously recording graphically the steam pressure and the piston displacement. This graphical record is called an indicator diagram. The area of the diagram (Fig. 203) is the net work done on one side of the piston during a working stroke and a return stroke.

94. Unit of work. The unit of work is the work performed by a unit force acting through a unit distance in the direction of the force.

The unit of work used in engineering calculations is the *foot-pound*, or the amount of work done in lifting a weight of one pound through a vertical distance of one foot. This unit varies

slightly with the location on the earth's surface, but for practical purposes it is considered constant.

95. Power. Power is the rate at which work is performed. The element of time does not enter into the definition of work, so that a definite amount of work is the same whether performed quickly or slowly. Among several machines the one which performs a given amount of work in the shortest time is said to be the most powerful machine. The usual unit of power is the horse power. Work is done at the rate of one horse power when 33,000 foot-pounds of work are done in one minute.

The unit of power used in electrical engineering is called the kilowatt. The relation between the kilowatt and horse power is

$$1 \text{ H.P.} = 0.746 \text{ kilowatt.}$$

EXAMPLE

Find the horse power of an engine which hauls a train weighing 1000 T. at a speed of 15 mi. per hour if the total resistance to the motion of the train amounts to 20 lb. per ton.

Solution. The total resistance is $1000 \times 20 = 20{,}000$ lb. The distance passed over in one second is 22 ft. Hence the work done per second is 440,000 ft.-lb., and the horse power is

$$\frac{440{,}000}{550} = 800 \text{ H.P.}$$

96. Work of a resultant force. If the point of application of several concurrent forces is displaced, and if during this displacement the several forces and their resultant maintain their magnitudes and directions, then the *work done by the resultant is equal to the sum of the work done by the several forces.* This follows at once from the fact that the component in a given direction of the resultant of any number of concurrent forces is equal to the sum of the components in that direction of the separate forces.

EXAMPLES

1. Show that the work done by the resultant R, Fig. 204, in moving the particle O a distance S along the line MN is equal to the work done by the rectangular components of R.

Solution. By definition the work of R is $R \cos (\theta - \alpha) \cdot S$. The work of X is $X \cos \alpha \cdot S$, and the work of Y is $Y \sin \alpha \cdot S$.

But $$X = R \cos \theta \quad \text{and} \quad Y = R \sin \theta,$$

and hence the work of the rectangular components becomes

$$X \cos \alpha \cdot S + Y \sin \alpha \cdot S = R \cos \theta \cos \alpha \cdot S + R \sin \theta \sin \alpha \cdot S = R \cos (\theta - \alpha) \cdot S.$$

2. In Fig. 204 show that when the particle is displaced a distance ds along a curve to which the line of action of R is tangent at O the work $R\,ds$ done by R is equal to $X\,dx + Y\,dy$.

Solution. Since

$$R = X \cos\theta + Y \sin\theta,$$

$$R\,ds = X \cos\theta \cdot ds + Y \sin\theta \cdot ds.$$

Therefore

$$R\,ds = X\,dx + Y\,dy.$$

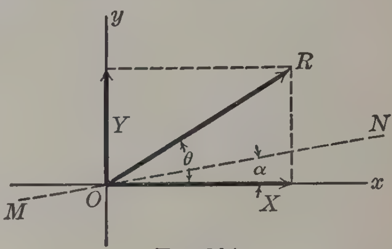

FIG. 204

97. Work of a constant couple. Given two equal and opposite forces, F and F', forming a couple whose moment is Fl, acting upon a body; to determine the work done by the forces when the body undergoes (a) translation, (b) rotation.

Translation. Let the body be translated a distance AM in a direction making an angle ϕ with the direction of the force F. By § 92 the work done by the force F is $F \cos\phi \cdot \overline{AM}$ and the work done by F' is $F' \cos(180 - \phi) \cdot \overline{AM}$. Hence the total work is zero.

Rotation. Since a couple can be moved anywhere in its plane without altering its effect, the mid-point O of the couple arm can be brought into coincidence with the center of rotation. When the forces F and F' remain constant and perpendicular to the arm during the rotation, the work done by each force F is F times the length of the circular arc described by its point of application. Hence the total work of the couple is

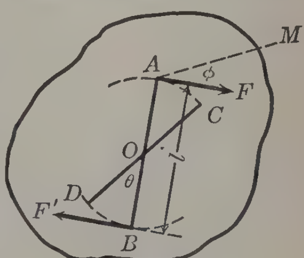

FIG. 205

$$W = \frac{Fl}{2} \cdot \theta + \frac{Fl}{2} \cdot \theta = Fl\theta.$$

Hence *the work of a constant couple is the product of the moment of the couple and the angular displacement measured in radians.*

The work of a variable couple may be obtained by finding the sum of the work done by each of the forces of the couple. The work of each force, in general, involves a line integral as in the general case of § 92.

98. Work of equivalent systems of forces. Any system of forces in a plane can be reduced to a single force or a couple (§ 47). It follows, therefore, that two equivalent systems of forces must reduce to the *same* force or couple. Hence in the same displacement the work done by each of two equivalent systems of forces is the same.

In computing the total work of a couple and a force whose line of action lies in the plane of the couple, considerable simplification may be effected by first reducing the couple and force to a single force, as in § 35.

99. Work done in raising a particle against gravity. The force necessary to move a particle of weight w upward along a path, an element ds of which makes an angle θ with the vertical, is $w \cos \theta$.

The work done by the force in moving the particle upward along the path ds is $w \cos \theta \cdot ds$. This may be written $w \cdot ds \cos \theta$. The vertical distance the particle has been raised is $ds \cdot \cos \theta$. Hence the work done by the force against gravity is the weight w multiplied by the *vertical distance* through which the particle has been raised.

The total work done by the force in raising a particle against gravity along any curve from a point whose ordinate is y to a point whose ordinate is y' is

$$\int_s^{s'} w \cos \theta \, ds = w \int_s^{s'} \cos \theta \, ds = w \int_y^{y'} dy = w(y' - y).$$

Therefore the work done in raising a particle of weight w from a point whose ordinate is y to any other point whose ordinate is y', *along any path whatsoever*, is equal to the weight multiplied by the vertical distance through which the body is raised.

100. Total work of raising a system of particles or bodies against gravity. The total work of raising a system composed of parts whose weights are w_1, w_2, w_3, w_4, etc. and whose initial heights or ordinates are y_1, y_2, y_3, y_4, etc. and whose final ordinates are y'_1, y'_2, y'_3, y'_4, etc. is given by

$$\begin{aligned}
\text{Work} &= w_1(y'_1 - y_1) + w_2(y'_2 - y_2) + w_3(y'_3 - y_3) + \cdots \\
&= (w_1 y'_1 + w_2 y'_2 + w_3 y'_3 + \cdots) - (w_1 y_1 + w_2 y_2 + w_3 y_3 + \cdots) \\
&= (w_1 + w_2 + w_3 + \cdots)\bar{y}' - (w_1 + w_2 + w_3 + \cdots)\bar{y} \\
&= (\bar{y}' - \bar{y})\Sigma w \text{ (by § 38).}
\end{aligned}$$

Hence the total work of raising a system of bodies against gravity is equal to the sum of the weights of the bodies multiplied by the vertical distance through which their center of gravity is raised.

101. Potential energy; field of force. Energy may be defined as the capacity of a body for doing work. When a body is capable of doing work it is said to possess energy. A body may possess energy owing to its elasticity, chemical composition, magnetic property, temperature, motion, position, etc. A body is said to have *potential energy* when it is capable of doing work by virtue of its position. A weight of 20 pounds placed 4 feet above a floor has 80 foot-pounds of potential energy with respect to the floor.

A region in which a free body moves with an acceleration is called a *field of force*. Familiar examples of a field of force are the gravitational field of force of the earth and a magnetic field of force near a magnet. Over a small portion of the earth's surface and within small distances of its surface the field of gravitational force may be considered constant in magnitude and direction.

Under these circumstances the potential energy of a body of weight W which is h feet above an arbitrary horizontal plane is Wh foot-pounds. The force due to the gravitational field of the earth is of great importance in many problems in mechanics and engineering.

102. Conservative system of force. The work expended in overcoming a frictional force is dissipated in heat, sound, electrical phenomena, etc., and in general cannot be regained. On the contrary, the work done in raising a weight a distance against gravity may be wholly recovered after the lapse of any interval of time by permitting the weight to fall through the same distance. When the forces acting on a body are of such a nature that the work done in moving the body from an initial position to a final position is independent of the path which the body describes, the system of forces is called a *conservative system*. If forces exist in the system such that the work performed against them is not recoverable, the system is said to be *nonconservative*. In general, it is clear that forces whose direction of action reverses with a reversal of direction of motion do not form a conservative system.

PROBLEMS ON WORK

1. A body weighing 1 T. is pulled up a frictionless inclined plane 40 ft. long making an angle of 30° with the horizontal by a force parallel to the plane. Calculate the work done and show that it is the same as if the body were to be lifted vertically through a distance of 20 ft. *Ans.* 40,000 ft.-lb.

2. If the coefficient of friction between the body and the inclined plane of Problem 1 is 0.5, find the total work done in pulling the body up the inclined plane. Would the work done be the same if a shorter inclined plane, having the same height and the same coefficient of friction, were used? *Ans.* 74,640 ft.-lb.

3. How much work is done in raising the material of a well 40 ft. deep and 4 ft. in diameter if the material is piled on top of the ground in the form of a cone of semivertical angle 45°? The material weighs 110 lb. per cubic foot. *Ans.* 1,214,060 ft.-lb.

4. A passenger train consists of 10 coaches weighing 80 T. each. Find the work done by the locomotive in pulling the train over level track a distance of 1000 ft. if the train resistance is constant and equal to 15 lb. per ton. *Ans.* 12,000,000 ft.-lb.

5. A train coasts down a 1-per-cent grade a distance of 2000 ft. against a constant train resistance of 10 lb. per ton. How far will the train run on level track at the foot of the hill before coming to rest? *Ans.* 2000 ft.

6. The modulus of a spring is 60 lb. per inch. Find the maximum work stored in the spring if a weight of 240 lb. is attached to the lower end of the spring and slowly lowered. Find the maximum energy stored in the spring if the weight is attached to the unstretched spring and then dropped. How can the difference in the results be accounted for? *Ans.* 480 in.-lb., 1920 in.-lb.

7. Find the work done in moving a particle from the origin to the point (12, 0) if the force required at any point is (x^2+2) lb., x being measured in feet. *Ans.* 600 ft.-lb.

8. A coil spring having an unstressed length of 2 ft. is stretched by a slowly increasing force until its length is 2.5 ft., the tension at that time being 60 lb. How much work is done upon the spring? How much work must be expended in stretching the spring an additional 6 in.? *Ans.* 15 ft.-lb., 45 ft.-lb.

9. An automobile weighing 2 T. ascends a 2-per-cent grade at a speed of 30 mi. per hour. Air and frictional resistance amount to 50 lb. The engine speed is 800 R.P.M. What torque does the engine develop? *Ans.* 68.2 lb.-ft.

10. A chain 10 ft. long weighs 20 lb. per foot. One end is slowly lifted from the floor and raised 20 ft. Find the work done.
Ans. 3000 ft.-lb.

11. The piston of a steam hammer has an area of 50 sq. in., and the total weight of the hammer, piston rod, and piston is 800 lb. Find the total work performed by gravity and the steam pressure during a single down stroke if steam is admitted through one fourth of the stroke and allowed to expand through the remainder of the distance. The length of the stroke is 2 ft., and the boiler pressure is 100 lb. gauge. Neglect the clearance in the cylinder and assume that the steam pressure obeys the law $pv = k$. Atmospheric pressure $= 14.7$ lb. per square inch. *Ans.* 6972 ft.-lb.

12. A weight of 10 lb. falls a distance of 4 ft. and then compresses a spring 6 in. Find the modulus of the spring.
Ans. 30 lb. per inch.

13. A body B is placed between two compressed springs as shown in Fig. 206. The natural length of spring A is 6 in., its modulus is 100 lb. per inch, and its initial compression is 2 in. The natural length of spring C is 8 in.; its modulus is 40 lb. per inch. The frame D is adjusted so that the body B is in a position of equilibrium with spring A compressed 2 in. A variable force is applied to B in the line of the springs. How much work is done by this variable force when it reaches a maximum value of 80 lb.? *Ans.* $22\frac{6}{7}$ in.-lb.

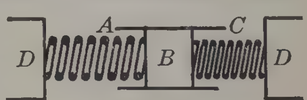

FIG. 206

14. A cable weighing 4 lb. per foot is suspended from a fixed point in the form of a loop, the other end passing over a winding drum. If the original length of the cable between the fixed point and the drum is 200 ft., find the work done against gravity in winding up 100 ft. of the cable. The winding drum and the fixed point are on the same level. *Ans.* 30,000 ft.-lb.

15. The diameter of the cylinder of a locomotive is 26 in., and the length of the stroke is 24 in. If the clearance volume is neglected, find the work done upon the piston during one stroke if steam is admitted during 6 in. at 200 lb. per square inch (boiler pressure) and the steam is then cut off and allowed to expand according to the law $pv = k$. *Ans.* 120,400 ft.-lb.

16. The area of an indicator card is found to be 4 sq. in., and the length of the card is 3 in. The scale of the indicator spring is 100 lb. per inch. The engine cylinder is 16 in. in diameter, and the stroke is 24 in. Find the work done per stroke of the piston.
Ans. 53,616 ft.-lb.

17. A rope whose length is 2π ft. is wrapped around a horizontal cylinder of radius 1 ft. The free end of the rope is on a horizontal diameter. If the cylinder is turned through 180° so that the free end of the rope descends a distance equal to one half the circumference, find the work of gravity. The rope weighs 2 lb. per foot.

Ans. 5.87 ft.-lb.

18. The earth attracts a body weighing 1 lb. at its surface with a force $\frac{a^2}{r^2}$, where a is the radius of the earth (4000 mi.) and r is the distance from the body to the center of the earth. Find the work done by this force while the body moves from an infinite distance to the surface of the earth. Does the result depend upon the path?

Ans. 21,120,000 ft.-lb.

19. A rope weighing 10 lb. per foot supports a load of 1000 lb. at its lower end. The rope passes upward 100 ft., then over a rough fixed horizontal cylinder ($\mu = 0.2$), thence horizontally to a winch. Find the work done by the winch in slowly raising the load 20 ft.

Ans. 52,026 ft.-lb.

PROBLEMS ON POWER

1. Find the total work in one horse-power hour.

Ans. 1,980,000 ft.-lb.

2. A ship of 10,000 H.P. travels 1000 mi. in 3 days. Find the average resistance to the motion of the ship if 80 per cent of the power is effective in driving the ship. *Ans.* 216,000 lb.

3. A locomotive exerts a drawbar pull of 8000 lb. at a speed of 25 mi. per hour. What will be the speed when it is working at the same rate if the drawbar pull decreases to 5000 lb.?

Ans. 40 mi. per hour.

4. Find the power developed by the locomotive of Problem 4, p. 143, if the train runs at uniform speed and travels a distance of 1000 ft. in 1 min. *Ans.* 363.6 H.P.

5. Find the power developed by the engine of Problem 15, p. 144, if the engine is double-acting and runs at 240 R.P.M.

Ans. 1750 H.P.

6. A constant torque of 100 lb.-ft. causes a shaft to turn at a uniform speed of 180 R.P.M. Find the power developed.

Ans. 3.43 H.P.

7. A shaft which transmits 100 H.P. turns at a speed of 360 R.P.M. Find the torque. *Ans.* 1458 lb.-ft.

8. A locomotive exerts a drawbar pull of 80,000 lb. at a speed of 7.5 mi. per hour. Find the drawbar horse power. *Ans.* 1600 H.P.

9. The head on a water turbine is 100 ft. The quantity of water passing through the turbine is 200 cu. ft. per second. The efficiency of the turbine is 80 per cent. Find the horse power developed. (Water weighs 62.5 lb. per cubic foot.) *Ans.* 1818 H.P.

10. The modulus of a spring is 2000 lb. per inch. The spring is stretched 1 ft., and the stored work is utilized to drive a machine which requires $\frac{1}{4}$ H.P. How long will the spring supply energy at the required rate? *Ans.* 87.3 sec.

11. What horse power will be required to draw a car at a speed of 20 mi. per hour up a 3-per-cent grade if the car weighs 25,000 lb. and the drawbar pull on level track is 250 lb.? *Ans.* 53.3 H.P.

103. Virtual work. Let a particle acted upon by several forces receive any infinitesimal displacement. The work done by all the forces is equal to the work done by the resultant of the forces. If the several forces are in equilibrium, the resultant force vanishes and the work done by the forces acting upon the particle during any infinitesimal displacement is zero. It is assumed that each of the forces remains sensibly constant in magnitude and direction during the displacement. In order that the forces may remain sensibly constant in magnitude and direction, the displacement is usually made an *infinitesimal* displacement. The infinitesimal displacement may be an actual displacement or any assumed or hypothetical displacement consistent with the constraints. Such arbitrary infinitesimal displacements are called *virtual* displacements. The work done by any one of the forces during a virtual displacement is called the *virtual work* of that force.

The principle of virtual work for a system of forces acting on a particle may be stated as follows: *If a system of forces acting upon a particle is in equilibrium, the virtual work of the system of forces is zero for all virtual displacements.*

Conversely, *if the virtual work of a system of forces acting on a particle is zero for all virtual displacements, the system of forces is in equilibrium.* For, if the system of forces is not in equilibrium it may be replaced by a resultant force. The work of this resultant would not be zero for *any* arbitrary displacement. This contradicts the hypothesis, and hence the resultant must be zero and the system of forces is in equilibrium.

By the simple process of addition the principle of virtual work can be extended to any *system* of particles.

Since the total *internal* work is zero for a displacement which does not alter the internal configuration of a rigid body, the internal forces do not appear in the equation of virtual work for a rigid body. The principle of virtual work stated above for a particle therefore holds for any rigid body.

104. Forces which do no work. The process of writing the equations of virtual work is facilitated by noticing that the following external forces do no work when acting on a rigid body which is subject to certain constraints:

1. If two particles of a system act upon each other by means of forces along the straight line joining them, and if the distance between the particles remains invariable for any displacement, the sum of the virtual work of the action and reaction is zero.

2. If any body of the system is constrained to turn around a point or axis fixed in space, the virtual work of the reaction of the point or axis is zero.

3. If any point of a body is constrained to slide on a *smooth* surface fixed in space, the virtual work of the reaction of the surface is zero.

4. If any body rolls without sliding on a surface fixed in space, the work of the reaction is zero.

5. If two bodies of a system roll on each other, the sum of the virtual work of the action and reaction is zero.

A method of applying the principle of virtual work is as follows:

1. Select a *fixed* set of coördinate axes.

2. *Assume* that the virtual displacements are each *positive*, whether possible or not.

If the angle between a force and the assumed positive displacement of its point of application is less than 90°, the work is positive; if the angle is greater than 90°, the work is negative.

3. Equate the algebraic sum of the virtual work of the external forces to zero.

4. Write the geometric relation between the variable coördinates and differentiate it to obtain a relation between the virtual displacements. Sometimes a relation between the virtual displacements may be obtained by inspection.

5. A relation between the forces may be obtained by eliminating the virtual displacements.

EXAMPLES

1. Find the force T necessary to hold the weight W in equilibrium in the system of pulleys shown diagrammatically in Fig. 207.

Solution. Select the axes as shown in the figure. Let y and y' be the ordinates of the points of application of the force T and the weight W respectively. If the virtual displacements are assumed positive, the work done by the force T is $T\,dy$ and the work done by the weight W is $-W\,dy'$. Hence, by the principle of virtual work,

$$T\,dy - W\,dy' = 0. \qquad (1)$$

It is evident from the geometric configuration that

$$dy = 5\,dy'. \qquad (2)$$

From (1) and (2), $\quad T = \dfrac{W}{5}.$

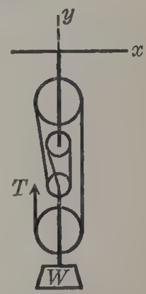

FIG. 207

2. Find the relation between the force P and the weights W and w in the toggle joint shown in Fig. 208.

Solution. If the axes are selected as shown and the coördinates of the point of application of P are x and y, the ordinate of W is $2\,y$ and the ordinates of the centers of gravity of the rods are $\dfrac{y}{2}$ and $\dfrac{3\,y}{2}$ respectively.

If the virtual displacements are each assumed positive, the virtual work of the force P is $-P\,dx$, the virtual work of the weight W is $-W\,2\,dy$, the virtual work of the upper rod is $-w\,\frac{3}{2}\,dy$, and the virtual work of the weight of the lower rod is $-w\,\frac{1}{2}\,dy$. Hence

$$-P\,dx - 2\,W\,dy - 2\,w\,dy = 0. \qquad (1)$$

The geometric relation is

$$x^2 + y^2 = l^2. \qquad (2)$$

From (2), $\quad dx = -\dfrac{y}{x}\,dy. \qquad (3)$

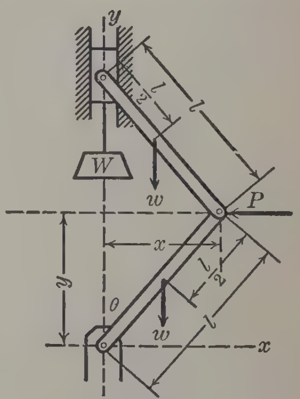

FIG. 208

From (1) and (3), $\quad P = 2(W + w)\dfrac{x}{y} = 2(W + w)\tan\theta.$

PROBLEMS

1. Find the force exerted by a hydraulic press if the depression of the handle of the pump 1 in. by a force of 10 lb. causes the piston of the press to rise 0.001 in. Neglect friction. *Ans.* 10,000 lb.

2. An endless chain of weight 10 lb. rests in the form of a horizontal ring on the surface of a smooth sphere. The radius of the ring is 2 ft., and the radius of the sphere is 4 ft. Find the tension in the chain due to its weight. *Ans.* 0.92 lb.

3. The wedge A supporting a weight of 100 lb. is held in equilibrium by a force P, as shown in Fig. 209. Find the force P. Neglect friction. *Ans.* 500 lb.

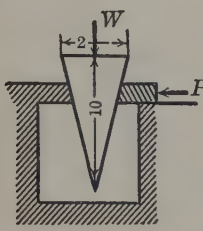

FIG. 209

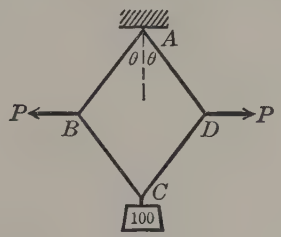

FIG. 210

4. Four rods of negligible weight and equal length form a rhombus, which is supported at A and which carries a load of 100 lb. at C, as shown in Fig. 210. Find the forces P required to maintain equilibrium when θ is 30°. *Ans.* 57.7 lb.

5. Solve Problem 4 if each rod weighs 25 lb. *Ans.* 86.6 lb.

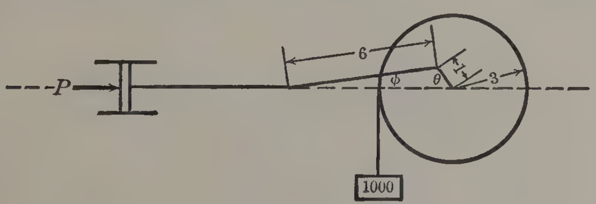

FIG. 211

6. Find the force P which must be applied to the piston of the steam engine shown in Fig. 211 to produce a couple of 3000 lb.-ft. if the crank angle θ is 60°. Disregard friction. *Ans.* 3195 lb.

7. A weight of 10 lb. rests on a smooth plane inclined at an angle of 15° with the horizontal. It is held in equilibrium by the force P, making an angle of 30° with the plane. Find P. *Ans.* 2.99 lb.

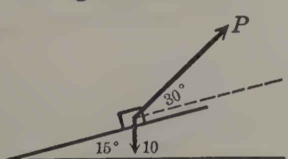

FIG. 212

8. A rod OA, 10 ft. long and of negligible weight, carries a load of 100 lb. at its center. Find the weight P required to hold the rod in equilibrium at an angle of 30° with the vertical if the distance OB is 15 ft. *Ans.* 26.92 lb.

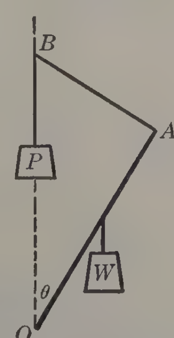

FIG. 213

9. Four rods are jointed at the ends to form a rectangle $ABCD$. The side AB is fixed in a vertical position, and the remaining rods are held in position by a string joining the middle points of CD and BC. The rods AB, BC, CD, and DA weigh 10 lb., 20 lb., 10 lb., and 20 lb. respectively. Find the tension in the string. *Ans.* 134.2 lb.

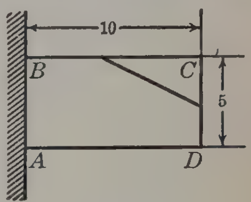

FIG. 214

10. A tripod, whose weight may be neglected, rests upon a smooth floor. The legs are kept in position by a string around their lower ends. Find the tension in the string if a weight of 50 lb. is suspended from the upper joint. The legs make angles of 60° with the floor. *Ans.* 5.55 lb.

11. A triangle ABC consists of three equal rods weighing 4 lb. each. Find the horizontal thrust in BC if the triangle is suspended from the vertex A. *Ans.* 2.31 lb.

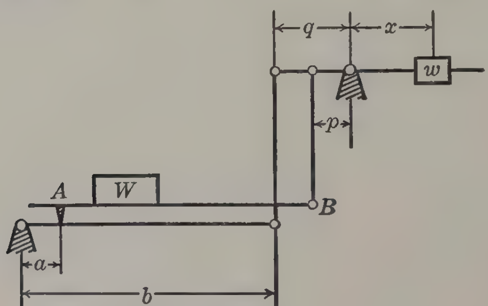

FIG. 215

12. Show that the weighing machine of Fig. 215 will indicate the same weight $W = \frac{wx}{p}$ of the body W regardless of the position of W on the platform AB if $aq = bp$.

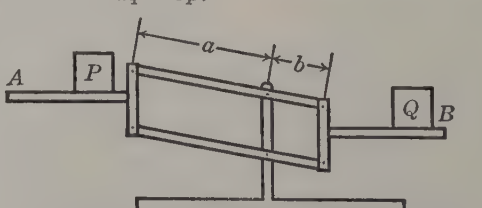

FIG. 216

13. The Roberval balance shown in Fig. 216 is in equilibrium when the weights P and Q are removed. Show that there is equilibrium if $Pa = Qb$ regardless of the position of the weights P and Q on the pans A and B.

105. Work function. Let a particle be constrained to move along a plane curve, and let R be the resultant of all the forces acting upon the particle, including the reaction of the curve upon the particle. Let the resultant force R be resolved into any two rectangular components X and Y, and let (x, y) be the coördinates of the particle. The work done by the resultant R or its components X and Y on the particle when it undergoes a small displacement ds along the curve is, by Example 2, p. 140,

$$W = \int R\,ds = \int (X\,dx + Y\,dy). \tag{1}$$

This integral may be calculated when X and Y are functions of x and y only and the path of the moving particle is given. For, from the equation of the path, y can be expressed in terms of x, and hence X and Y can be expressed in terms of x. Likewise dy can be expressed in terms of x and dx. Thus the integral is expressed in terms of the single variable x, and it can be evaluated in many cases. In general, the value of the integral from a point (x_1, y_1) to a point (x_2, y_2) will vary with the path, as is evident when frictional forces are considered.

If, however, as happens in many cases, $X\,dx + Y\,dy$ is an exact differential, it can be integrated, *without knowing the equation of the path*, from the point (x_1, y_1) to (x_2, y_2), giving

$$\int (X\,dx + Y\,dy) = U, \tag{2}$$

where U is a function of the independent variables x and y.

Differentiated totally, (2) gives

$$X\,dx + Y\,dy = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy,$$

and, since x and y are independent variables,

$$X = \frac{\partial U}{\partial x} \quad \text{and} \quad Y = \frac{\partial U}{\partial y}. \tag{3}$$

The condition that $X\,dx + Y\,dy$ is an exact differential is

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x},$$

since each is equal to $\dfrac{\partial^2 U}{\partial x\,\partial y}$. (4)

If such a function U exists, it is called a force function. It is called a force function because the components of the force acting may be obtained from it by partial differentiation; that is,

$$\frac{\partial U}{\partial x} = X \quad \text{and} \quad \frac{\partial U}{\partial y} = Y. \tag{5}$$

The system of forces X and Y for which a force function U exists (that is, for which $X\,dx + Y\,dy$ is an exact differential) is called a *conservative system.*

Any set of values x_0 and y_0 of x and y may be chosen as the standard position; the value of W therefore becomes a function of the upper limit x and y.

$$W = \int (X\,dx + Y\,dy) = U(x, y) - U(x_0, y_0).$$

If the upper limit is fixed as the standard position and the lower limit remains variable, then

$$W = \int (X\,dx + Y\,dy) = U(x_0, y_0) - U(x, y). \tag{6}$$

The last expression is called the *potential energy* of the forces referred to the position (x_0, y_0). When the potential energy of the standard position is arbitrarily taken as zero,

$$W = -U. \tag{7}$$

106. Conditions of equilibrium; energy method. When a body is in equilibrium under the action of any system of forces, the work done in any (that is, every) infinitesimal displacement consistent with the constraints is zero. Analytically this is expressed for two dimensions by

$$\frac{\partial W}{\partial x} = 0, \quad \frac{\partial W}{\partial y} = 0, \quad \frac{\partial W}{\partial \theta} = 0, \tag{1}$$

where x, y, and θ are independent coördinates which specify the position of the body.

But these conditions are exactly the condition for a maximum or minimum.

To determine the position of equilibrium, *first* the work function is expressed as a function of the coördinates, and *second* the usual methods of calculus are used to find the values of x, y, and θ which make W a maximum or a minimum. If x, y,

and θ are independent, the partial derivative of W with respect to each variable is equated to zero. This is equivalent to choosing the virtual displacement so that only one of the variables changes at a time.

Since the work is equal to the negative potential energy (§ 105), the potential energy U may replace W in (1). Hence it follows that if the potential energy of a system is a maximum or a minimum, the system is in equilibrium.

107. Stable and unstable equilibrium. If the forces which are brought into action when a body is slightly displaced from its equilibrium position are such as tend to move the body back into its former position, the equilibrium is stable. If the forces tend to continue the motion, the equilibrium is unstable. If the potential energy is constant for all positions, the equilibrium is neutral.

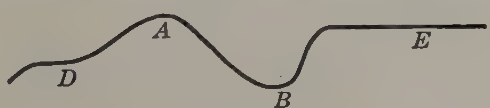

FIG. 217

A heavy bead sliding on the smooth wire shown in Fig. 217 is in unstable equilibrium at A, in stable equilibrium at B, and in neutral equilibrium at E.

108. Criterion of stability; one degree of freedom. Let x be a variable whose value determines the position of the body. If the system is assumed to be a conservative system, the work function or the potential energy is a function of the single coordinate x. Suppose the body to be in equilibrium at a point $x = x_0$. Then $W = f(x)$ may be expanded in the neighborhood of the equilibrium position x_0, giving

$$W = W_{x_0} + x\left(\frac{\partial W}{\partial x}\right)_{x_0} + \frac{x^2}{\lfloor 2}\left(\frac{\partial^2 W}{\partial x^2}\right)_{x_0} + \frac{x^3}{\lfloor 3}\left(\frac{\partial^3 W}{\partial x^3}\right)_{x_0} + \cdots.$$

Since the body is in equilibrium at $x = x_0$,

$$\left(\frac{\partial W}{\partial x}\right)_{x_0} = 0.$$

For a position near x_0, x is small, and consequently terms containing higher powers of x may be neglected, giving

$$W - W_{x_0} = \frac{x^2}{2}\left(\frac{\partial^2 W}{\partial x^2}\right)_{x_0}.$$

From this equation it follows that if $\left(\frac{\partial^2 W}{\partial x^2}\right)_{x_0}$ is positive, $W - W_{x_0}$ is positive for small positive or negative values of x and hence the value W_{x_0} is a *minimum*. Similarly if $\left(\frac{\partial^2 W}{\partial x^2}\right)_{x_0}$ is negative, W_{x_0} is a *maximum*. If $\left(\frac{\partial^2 W}{\partial x^2}\right)_{x_0}$ is zero, further terms in the expansion must be considered in a similar manner.

EXAMPLE

A right cone is suspended from a smooth vertical wall by a string of length equal to the diameter of the base of the cone. The string is attached to a point in the circumference of the base, and the cone is in equilibrium with a point of its base in contact with the wall. Find the angle θ which the string makes with the wall, and determine whether or not the equilibrium is stable.

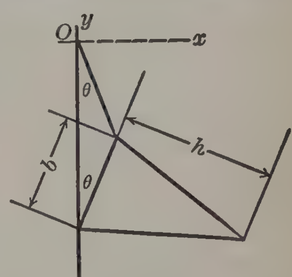

FIG. 218

Solution. Let the origin be taken where the string is attached to the wall. The ordinate of the center of gravity of the cone is

$$y = -\frac{3}{2} b \cos\theta - \frac{h}{4}\sin\theta.$$

By § 106 the position of equilibrium is determined by equating $\frac{dy}{d\theta}$ to zero. Hence

$$\frac{dy}{d\theta} = \frac{3}{2} b \sin\theta - \frac{h}{4}\cos\theta = 0,$$

and therefore

$$\tan\theta = \frac{h}{6\,b}.$$

The question of stability is determined by examining the sign of $\frac{d^2y}{d\theta^2}$.

$$\frac{d^2y}{d\theta^2} = \frac{3}{2} b \cos\theta + \frac{h}{4}\sin\theta.$$

From the geometry of the figure, the angle θ cannot be greater than 90°, and therefore the second derivative is positive and the cone is in stable equilibrium.

PROBLEMS

1. The center of gravity of a sphere is not at its geometric center. Apply the energy criterion to distinguish between the positions of stable and unstable equilibrium.

2. One end of a rope is attached to a small ring. The other end is passed around a square horizontal beam of side 8 in., through the ring, and then attached to a weight of 50 lb. Neglecting friction, find the angle which the rope makes with the lower side of the beam. *Ans.* 30°.

3. A solid of uniform density consists of a hemispherical base of radius 6 in. and a right cylinder, as shown in Fig. 219. Find the greatest permissible length of the cylindrical portion in order that the solid may rest in an upright position. *Ans.* $h = 3\sqrt{2}$ in.

FIG. 219

4. A rod 16 ft. long rests on a smooth peg at A and against a smooth wall at C, as shown in Fig. 220. The distance from the peg to the wall being 1 ft., find the angle θ necessary for equilibrium. *Ans.* $\theta = 30°$; unstable.

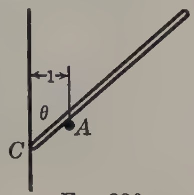

FIG. 220

5. A square board is suspended from a smooth vertical wall in a plane at right angles to the wall by a string having one end attached to one of its corners. The length of the string is equal to the length of a side of the square. Find the position of equilibrium and examine the question of stability.

Ans. Tangent of angle between wall and string $= \frac{1}{3}$; stable.

6. A solid cone of altitude 12 in. and semivertical angle 15° is placed vertex downward in a hole of radius x in. Show that the equilibrium in the vertical position is stable or unstable depending upon whether $8x$ is greater or less than 9 in.

CHAPTER XII

MOMENT OF INERTIA

109. Moment of inertia. *The sum of the products obtained by multiplying the mass of every particle of a body by the square of its distance from a point, line, or plane is called the moment of inertia of the body with respect to the point, line, or plane respectively.* Although the moments of inertia of a body with respect to a *point* or a *plane* are not directly used in mechanics, they are of great utility in finding the moment of inertia of the body with respect to a *line*. Summations leading to the moment of inertia of a body with respect to a line or axis occur very often in mechanical problems, and it is therefore convenient to collect them into a single chapter for reference.

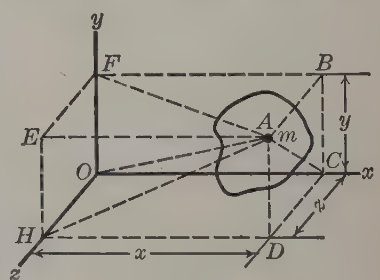

FIG. 221

The radius of gyration of a body with respect to an axis is defined as the distance from the axis to such a point that if the mass of the body were concentrated in a particle at that point, the moment of inertia of the particle would be the same as the moment of inertia of the original body. Thus if M is the mass of a body and k is the radius of gyration of the body with respect to an axis, the moment of inertia I about that axis may be expressed by

$$I = Mk^2, \tag{1}$$

from which

$$k = \sqrt{\frac{I}{M}}. \tag{2}$$

Let m be the mass of a particle of a body, and let its coördinates be x, y, and z.

The moments of inertia of the particle of mass m with respect to the coördinate *planes* yz, xz, and xy are, by definition,

$$mx^2, \quad my^2, \quad \text{and} \quad mz^2,$$

and the moments of inertia of the whole body with respect to the same planes are

$$\Sigma(mx^2),\ \Sigma(my^2),\ \text{and}\ \Sigma(mz^2). \tag{3}$$

The moments of inertia of the particle with respect to the *axes* of x, y, and z are, by definition,

$$m\overline{AC}^2,\quad m\overline{AF}^2,\quad \text{and}\quad m\overline{AH}^2,$$

where

$$\overline{AC}^2 = y^2 + z^2,\quad \overline{AF}^2 = x^2 + z^2,\quad \text{and}\quad \overline{AH}^2 = x^2 + y^2.$$

The moments of inertia of the body with respect to the x, y, and z axes are, therefore,

$$\Sigma[m(y^2 + z^2)],\ \Sigma[m(x^2 + z^2)],\ \text{and}\ \Sigma[m(x^2 + y^2)]. \tag{4}$$

The moment of inertia of the particle with respect to the *point* O is, by definition,

$$m\overline{AO}^2,$$

where $\overline{AO}^2 = x^2 + y^2 + z^2$.

The moment of inertia of the body with respect to the point O is, therefore,

$$\Sigma[m(x^2 + y^2 + z^2)]. \tag{5}$$

The relations between the quantities (3), (4), and (5) may be expressed by the following theorems:

110. Fundamental relations. *Theorem I. The moment of inertia of a body with respect to an axis is equal to the sum of the moments of inertia of the body with respect to any two planes at right angles which intersect on the axis.*

This theorem may be established by adding any two of the quantities (3) to obtain the quantities (4).

Theorem II. The moment of inertia of a body with respect to a point is equal to the sum of the moments of inertia of the body with respect to three rectangular planes which intersect at the point.

This theorem is established by adding the three quantities (3) to obtain the quantity (5).

Theorem III. The moment of inertia of a body with respect to a point is equal to one half of the sum of the moments of inertia with respect to three rectangular axes which intersect in the point.

This theorem is established by comparing the sum of the quantities (4) with the quantity (5).

The use of Theorems I, II, and III is illustrated in the following examples involving simple geometric solids which are of common occurrence in applied mechanics.

EXAMPLES

1. Find the moment of inertia of a thin spherical shell with respect to any diameter.

Solution. Let m be the mass of the shell and let r be its radius. Since all parts of the shell are equidistant from the center, the moment of inertia of the shell with respect to the center is mr^2. From symmetry the moment of inertia of the shell is the same with respect to all diameters and with respect to all planes passing through the center. By Theorem II, § 110, the moment of inertia of the shell with respect to a diametral plane is $\frac{mr^2}{3}$ and by Theorem I, the required moment of inertia is

$$I = \tfrac{2}{3}\, mr^2.$$

2. Determine the moment of inertia of a sphere of mass M and radius R with respect to any diameter (Fig. 222).

Solution. Let the density of the sphere be γ. The mass of a thin shell of radius r and thickness dr is $\gamma\, 4\, \pi r^2\, dr$. The moment of inertia of this shell with respect to the center is

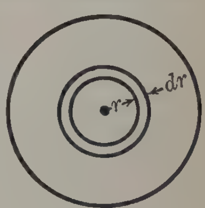

FIG. 222

$$\gamma\, 4\, \pi \int_0^R r^4\, dr = \frac{\gamma\, 4\, \pi R^5}{5} = \tfrac{3}{5}\, MR^2,$$

since $M = \gamma\, \frac{4}{3}\, \pi R^3$.

From considerations of symmetry it follows that the moments of inertia with respect to all diameters of the sphere are the same. By Theorem III, § 110, it follows that the moment of inertia of a sphere with respect to a diameter is

$$I = \tfrac{2}{3} \cdot \tfrac{3}{5}\, MR^2 = \tfrac{2}{5}\, MR^2.$$

3. Find the moment of inertia of a right cylinder of mass M and radius R with respect to its longitudinal axis.

Solution. The moment of inertia of a thin cylindrical shell of radius r and thickness dr is $\gamma\, 2\, \pi rh \cdot r^2\, dr$, and the moment of inertia of the cylinder is

$$I = \gamma\, 2\, \pi h \int_0^R r^3\, dr = \frac{\gamma \pi h R^4}{2}.$$

Also, since $M = \gamma \pi R^2 h$, the moment of inertia of the cylinder is

$$I = \frac{MR^2}{2}.$$

4. Find the moment of inertia of a right cone of radius R and height h, Fig. 223, with respect to its axis.

Solution. The moment of inertia of a thin cylindrical shell of radius x, thickness dx, and height y is $\gamma\, 2\,\pi xy \cdot x^2\, dx$. From the geometry of the figure,

$$\frac{x}{R} = \frac{h-y}{h}, \quad \text{or} \quad y = \frac{h}{R}(R-x).$$

The moment of inertia of the cone is, therefore,

$$I = \frac{\gamma\, 2\,\pi h}{R}\int_0^R (R-x)x^3\, dx = \frac{\gamma \pi h R^4}{10}.$$

Also, since $M = \gamma\, \frac{1}{3}\, \pi R^2 h$,

$$I = \tfrac{3}{10}\, MR^2.$$

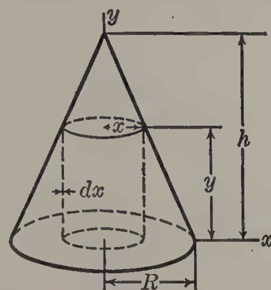

FIG. 223

5. Find the moment of inertia of a rectangular block of mass M with sides of length a, b, and c with respect to a central axis parallel to an edge of length c, Fig. 224.

Solution. The moment of inertia of a thin plate of height b and width c with respect to the yz plane is $\gamma bcx^2\, dx$. The moment of inertia of the whole block with respect to the yz plane is

$$\gamma\, 2\, bc \int_0^{\frac{a}{2}} x^2\, dx = \frac{\gamma a^3 bc}{12} = \frac{Ma^2}{12}.$$

Similarly the moment of inertia with respect to the xz plane is $\dfrac{Mb^2}{12}$.

Hence, by Theorem I, § 110, the required moment of inertia is

$$I = \frac{M}{12}(a^2 + b^2).$$

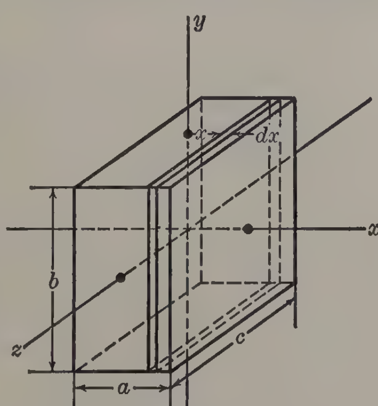

FIG. 224

PROBLEMS

1. Find the moment of inertia of a thin rod of length l and mass M with respect to an axis perpendicular to the rod and passing through one of its ends. *Ans.* $\frac{1}{3}\, Ml^2$.

2. Make use of Problem 1 to show that the moment of inertia of a right circular cylinder of mass M, radius R, and height h with respect to the plane of either base is $\frac{1}{3}\, Mh^2$. Is this result valid for a hollow cylinder when M is the mass of the hollow cylinder?

3. Make use of Problem 1 to show that the moment of inertia of a rectangular parallelepiped of mass M, having sides of length a, b, and c, with respect to the plane containing the edges of length a and b is $\frac{1}{3}\, Mc^2$.

4. From Example 3, p. 158, and Theorem I, § 110, show that the moment of inertia of a right circular cylinder of mass M and radius R with respect to any plane passing through its longitudinal axis is $I = \frac{1}{4} MR^2$.

5. Show that the moment of inertia of a thin hollow cylinder of mass M and radius r with respect to a plane containing the longitudinal axis is $\frac{1}{2} Mr^2$.

6. Combine the results of Problems 2 and 5 by Theorem I, § 110, to show that the moment of inertia of a thin hollow circular cylinder with respect to a diameter of either base is $I = M\left(\frac{r^2}{2} + \frac{h^2}{3}\right)$.

7. Combine the moments of inertia of Problems 2 and 4 to show that the moment of inertia of a right circular cylinder of mass M, radius R, and height h with respect to a diameter of the base is $M\left(\frac{R^2}{4} + \frac{h^2}{3}\right)$.

8. Combine the moments of inertia of Problem 3 and Example 5, p. 159, to show that the moment of inertia of a rectangular block of sides a, b, and c with respect to a central axis in the ab face parallel to b is $I = M\left(\frac{a^2}{12} + \frac{c^2}{3}\right)$.

9. Find the moment of inertia of a right circular cone of mass M, radius R, and height h, with respect to the plane of the base.

HINT. From Problem 1 and Example 4, p. 159, the required moment of inertia is

$$\gamma \tfrac{2}{3} \pi \int_0^R xy^3\, dx, \quad \text{where} \quad y = \frac{h(R-x)}{R}.$$

Ans. $\frac{M}{10} h^2$.

10. From Problem 9 and Example 4, p. 159, show that the moment of inertia of a right circular cone of mass M, radius R, and height h with respect to a diameter of the base is $\frac{M}{10}(h^2 + \frac{3}{2} R^2)$.

11. Find the moment of inertia of an ellipsoid of semiaxes a, b, and c with respect to the axis of a.

(For an elegant solution of this problem see Appell's "Traité de mécanique rationnelle," Vol. II, p. 4.)

Ans. $\frac{M}{5}(b^2 + c^2)$.

111. The product of inertia. *The sum of the products obtained by multiplying the mass of each particle of a body by the product of two of its coördinates is called the product of inertia of the body with respect to the axes of those coördinates.*

Thus in Fig. 225 the product of inertia of the particle of mass m situated at the point $A(x, y, z)$ with respect to the axes of x and y is

$$mxy,$$

and the product of inertia F of the body with respect to the same axes is

$$F = \Sigma(mxy). \qquad (1)$$

Likewise the products of inertia of the body with respect to the axes of y and z is

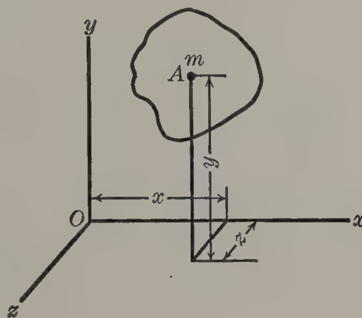

FIG. 225

$$D = \Sigma(myz), \qquad (2)$$

and with respect to the axes of z and x is

$$E = \Sigma(mzx). \qquad (3)$$

The product of inertia of a body, like the moment of inertia of a body with respect to a point or plane, is not of direct utility. It is principally employed in finding the moment of inertia of a body with respect to an axis.

EXAMPLE

Find the product of inertia of a rectangular block of sides a, b, and c with respect to the x and y axes (Fig. 226).

Solution. The product of inertia is

$$F = \int_0^a \int_0^b \int_0^c xy\,dx\,dy\,dz = \frac{Mab}{4}.$$

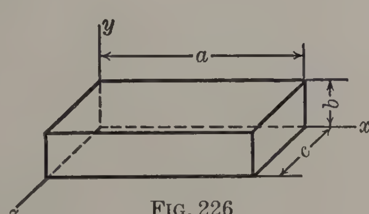

FIG. 226

PROBLEMS

1. Find the product of inertia of the triangular block of sides a, b, and c with respect to the x and z axes (Fig. 227). *Ans.* $\frac{1}{12}$ *Mab*.

2. Find the product of inertia of a slender rod of length l lying in the xy plane and making an angle of $(90° + \theta)$ with the positive x axis, the center of the rod being at the origin.

Ans. $-\dfrac{Ml^2}{12}\sin\theta\cos\theta.$

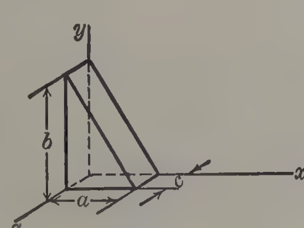

FIG. 227

112. The parallel-axis theorem. One of the most useful theorems relating to the subject of moment of inertia is the following:

The moment of inertia of a body with respect to any axis is equal to the moment of inertia of the body with respect to a parallel axis passing through its center of gravity plus the product of the total mass of the body by the square of the distance between the axes.

Proof. Let the moment of inertia of a body be referred to a given axis MN, Fig. 228. Let the y axis pass through the center of gravity G of the body, and let it be parallel to the given axis MN. Let the coördinates of the axis MN be $x = a$ and $z = c$. The moment of inertia with respect to the axis MN of any particle P of mass m, whose coördinates are x, z is evidently

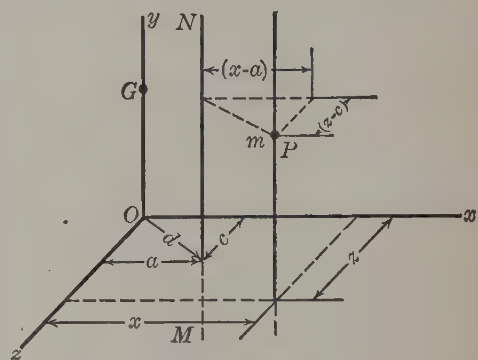

FIG. 228

$$m[(x-a)^2+(z-c)^2]=m[(x^2+z^2)+(a^2+c^2)]-2\,amx-2\,cmz,$$

and the moment of inertia of the body is

$$\Sigma\{m[(x^2+z^2)+(a^2+c^2)]\}-2\,a\Sigma(mx)-2\,c\Sigma(mz).$$

The last two terms vanish, since the axis of y passes through the center of gravity of the body (§ 37), and the moment of inertia of the body with respect to the axis MN becomes

$$I=\Sigma[m(x^2+z^2)]+\Sigma[m(a^2+c^2)].$$

The first term of this expression is the moment of inertia of the body with respect to an axis parallel to MN passing through the center of gravity, and the second term is the mass of the body multiplied by the square of the distance between the axes. Hence the theorem follows. If $\Sigma[m(x^2+z^2)]$ is replaced by I_G and $\Sigma[m(a^2+c^2)]$ is replaced by Md^2, the last equation may be written

$$I=I_G+Md^2. \tag{1}$$

From this expression it is clear that the moment of inertia for parallel axes in space will be least with respect to that axis which passes through the center of gravity.

In solving problems which involve a transfer of the moment of inertia from an axis to a second parallel axis neither of which passes through the center of gravity, the transfer is effected by first finding the moment of inertia with respect to a parallel axis through the center of gravity and then transferring from the axis through the center of gravity to the second axis. The parallel-axis theorem expressed by equation (1) is *not valid* for a transfer of the moment of inertia from one axis to a parallel axis if neither of the axes passes through the center of gravity.

EXAMPLES

1. Find the moment of inertia of a sphere with respect to an axis tangent to its surface.

Solution. The moment of inertia of a sphere with respect to a diameter is $\frac{2}{5}MR^2$, by Example 2, p. 158. Hence in (1), § 112,

$$I_G = \tfrac{2}{5}MR^2$$

and

$$I = \tfrac{2}{5}MR^2 + MR^2 = \tfrac{7}{5}MR^2.$$

2. Given three parallel axes, F, G, and H, the axis G passing through the center of gravity of a body whose mass is 5 slugs. The distance between the axes F and G is 2 ft., and the distance between G and H is 3 ft. The moment of inertia of the body with respect to the axis H is 65 slug-ft.-ft. Find the moment of inertia of the body with respect to the axis F.

Solution. From the parallel-axis theorem,

$$I_H = I_G + Md^2,$$

or

$$65 = I_G + 5 \times 9.$$

Hence

$$I_G = 20.$$

Also

$$I_F = I_G + Md_1^2,$$

or

$$I_F = 20 + 5 \times 4 = 40.$$

PROBLEMS

1. Find the moment of inertia of a right circular cylinder of mass M and radius R with respect to an axis lying in the cylindrical surface.

Ans. $\frac{3}{2}MR^2$.

2. Find the moment of inertia of a hemisphere of mass M and radius R with respect to a diameter of its base. Also find the moment of inertia with respect to a parallel axis through the center of gravity.

Ans. $\frac{2}{5}MR^2$, $\frac{83}{320}MR^2$.

3. Find the moment of inertia of a rectangular parallelepiped having sides of lengths a, b, and c with respect to an axis coinciding with an edge of length c. *Ans.* $\frac{1}{3}M(a^2 + b^2)$.

4. Make use of Problem 1, p. 159, to find the moment of inertia of a thin rod of mass M and length l with respect to an axis through the center of gravity perpendicular to the rod. *Ans.* $\frac{1}{12}Ml^2$.

5. Find the moment of inertia of a right circular cylinder of mass M and radius R with respect to a diameter of the circular section passing through the center of gravity. Make use of Problem 7, p. 160.

Ans. $M\left(\frac{R^2}{4} + \frac{h^2}{12}\right)$.

113. The parallel-axis theorem; product of inertia. *The product of inertia of a body with respect to any rectangular axes is equal to the product of inertia of the body with respect to parallel axes passing through its center of gravity plus the continued product of the total mass of the body and the two coördinates of the center of gravity referred to the other axes.*

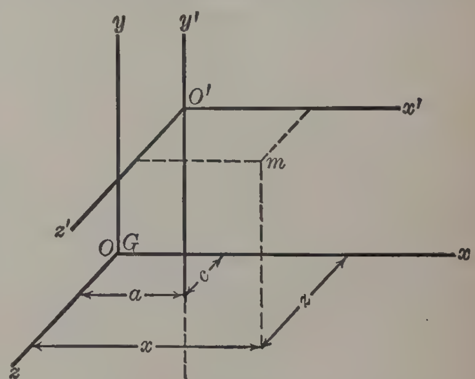

FIG. 229

Proof. Let the product of inertia of a body be referred to a set of axes x' and z' parallel to axes x and z, which pass through the center of gravity G of the body (Fig. 229). The product of inertia of the body referred to the axes x' and z' is $\Sigma(mx'z')$, and since $x' = x - a$ and $z' = z - c$, where a and c are the coördinates of the y' axis, the product of inertia is

$$F = \Sigma[m(x - a)(z - c)],$$

or

$$F = \Sigma(mxz) - c\Sigma(mx) - a\Sigma(mz) + \Sigma(mac).$$

The two middle terms vanish, since the y axis passes through the center of gravity, and hence

$$F = \Sigma(mxz) + \Sigma(mac). \tag{1}$$

The summation $\Sigma(mxz)$ is the product of inertia of the body with respect to axes through the center of gravity, and the

term $\Sigma(mac)$ is the product of the total mass of the body and the coördinates of the y' axis. It is evident that the theorem is true if G lies at any point on the y axis.

PROBLEMS

1. Solve the example on page 161 by means of the theorem of § 113.

2. Find the product of inertia of a cube of mass M and side of length l with respect to axes coinciding with adjacent edges.

Ans. $\dfrac{Ml^2}{4}$.

114. The moment of inertia of a body with respect to any axis passing through a point. Let it be required to find the moment of inertia of a body with respect to an axis OP passing through the point O, Fig. 230. Let the point O be selected as the origin of coördinates, and let α, β, and γ be the direction cosines of the axis OP about which the moment of inertia of the body is required. The moment of inertia of any particle of the body of mass m situated at a point Q, whose coördinates are x, y, z, is $m\overline{QR}^2$, where QR is the perpendicular distance from Q to the given axis OP.

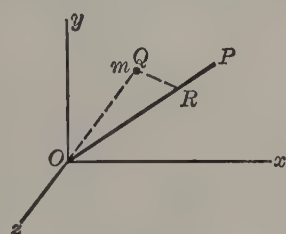

FIG. 230

From analytic geometry,

$$\overline{QR}^2 = \overline{OQ}^2 - \overline{OR}^2 \quad \text{and} \quad \overline{OQ}^2 = x^2 + y^2 + z^2.$$

Also, since OR is the projection of OQ on the axis OP,

$$OR = \alpha x + \beta y + \gamma z.$$

Hence $\quad \overline{QR}^2 = (x^2 + y^2 + z^2) - (\alpha x + \beta y + \gamma z)^2,$

and since $\quad \alpha^2 + \beta^2 + \gamma^2 = 1,$

$$\begin{aligned}\overline{QR}^2 &= (x^2 + y^2 + z^2)(\alpha^2 + \beta^2 + \gamma^2) - (\alpha x + \beta y + \gamma z)^2 \\ &= \alpha^2(y^2 + z^2) + \beta^2(z^2 + x^2) + \gamma^2(x^2 + y^2) - 2\,\beta\gamma yz \\ &\quad - 2\,\gamma\alpha zx - 2\,\alpha\beta xy.\end{aligned}$$

The moment of inertia of the entire body with respect to the axis OP is

$$\begin{aligned}\Sigma(m\overline{QR}^2) &= \alpha^2\,\Sigma[m(y^2 + z^2)] + \beta^2\,\Sigma[m(z^2 + x^2)] \\ &\quad + \gamma^2\,\Sigma[m(x^2 + y^2)] - 2\,\beta\gamma\Sigma(myz) - 2\,\gamma\alpha\Sigma(mzx) \\ &\quad - 2\,\alpha\beta\Sigma(mxy). \qquad (1)\end{aligned}$$

The first three terms on the right-hand side of (1) are the moments of inertia with respect to the x, y, and z axes multiplied by α^2, β^2, and γ^2 respectively ((4), § 109). The parts of the last three terms under the summation signs are the products of inertia of the body with respect to the coördinate axes ((1), (2), and (3), § 111). If A, B, and C are used to designate the moments of inertia with respect to the coördinate axes of x, y, and z respectively and if D, E, and F indicate the products of inertia of the body referred to the axes of y and z, z and x, and x and y respectively, the moment of inertia of the body with respect to the axis OP may be written

$$I = A\alpha^2 + B\beta^2 + C\gamma^2 - 2\,D\beta\gamma - 2\,E\gamma\alpha - 2\,F\alpha\beta. \qquad (2)$$

The *principal axes* of inertia at a given point are those axes with respect to which the products of inertia vanish. Hence if the principal axes are selected as axes of coördinates, (2) becomes

$$I = A\alpha^2 + B\beta^2 + C\gamma^2. \qquad (3)$$

It can be shown that the moment of inertia has its maximum, minimum, and intermediate values about the principal axes (§ 118). The principal axes of a body may often be determined by inspection. Thus, the principal diameters of an ellipsoid are principal axes.

PROBLEMS

1. Find the moment of inertia of a cube of side a and mass M with respect to a diagonal of one face.

HINT.
$$\alpha^2 = \beta^2 = \tfrac{1}{2} \text{ and } \gamma^2 = 0.$$
$$A = B = C = \tfrac{2}{3}\,Ma^2.$$

F is obtained from Problem 2, p. 165. *Ans.* $\frac{5}{12}\,Ma^2$.

2. Find the moment of inertia of a rectangular parallelepiped having edges a, b, and c with respect to one of its diagonals.

HINT. From Problem 3, p. 164,

$$C = \tfrac{1}{3}\,M(a^2 + b^2),\quad A = \tfrac{1}{3}\,M(b^2 + c^2),\quad B = \tfrac{1}{3}\,M(a^2 + c^2)$$

and from the example on page 161,

$$F = \tfrac{1}{4}\,Mab,\quad D = \tfrac{1}{4}\,Mbc,\quad E = \tfrac{1}{4}\,Mac.$$

Also $\alpha = \dfrac{a}{\sqrt{a^2+b^2+c^2}}$, $\beta = \dfrac{b}{\sqrt{a^2+b^2+c^2}}$, and $\gamma = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$.

Ans. $I = \dfrac{M}{6}\,\dfrac{b^2c^2 + a^2c^2 + a^2b^2}{a^2+b^2+c^2}$.

115. The moment of inertia of a plane area. The moment of inertia of a plane area with respect to a given axis may be defined as the sum of the products obtained by multiplying each element of the area by the square of its distance from the axis. On account of the analogy between the result of this summation and the summation leading to the moment of inertia of a body having mass, the former is called the moment of inertia of an area. The idea of mass is sometimes introduced into the moment of inertia of an area by regarding the area as a thin plate having mass. Problems which involve the so-called moment of inertia of area frequently occur in the mechanics of materials. Since there is no fundamental difference between the moment of inertia of solids and the moment of inertia of areas, the parallel-axis theorem is valid for areas as well as for solids.

116. Polar moment of inertia of a plane area. The polar moment of inertia of a plane area is the moment of inertia of the area with respect to an axis which is perpendicular to the plane of the area. The following theorem, which will be recognized as a special case of Theorem I of § 110, is often useful in finding the polar moment of inertia of a plane area.

Theorem. The polar moment of inertia of a plane area is equal to the sum of its moments of inertia about any two rectangular axes in its plane drawn from the point where the polar axis meets the plane.

EXAMPLES

1. Find the moment of inertia of a thin circular hoop of radius r and mass M with respect to a polar axis passing through its center.

Solution. Since the hoop is thin, all the particles of the hoop may be regarded as situated at a distance r from the center. The moment of inertia is, therefore,

$$I = Mr^2.$$

2. Find the moment of inertia of the area of a circular disk of radius R and area A with respect to its polar axis of symmetry.

Solution. Let the area be divided into thin rings of radius x and width dx concentric with its center. The area of an element is $2\pi x\,dx$ and its moment of inertia is $2\pi x^3\,dx$. The moment of inertia of the disk is, therefore,

$$2\pi \int_0^R x^3\,dx = \frac{\pi R^4}{2} = \frac{AR^2}{2}.$$

3. Find the moment of inertia of a circular disk with respect to a diameter in its plane.

Solution. From Example 2 and the theorem of § 116, the result is

$$\tfrac{1}{4} AR^2.$$

4. Find the moment of inertia of the area of a rectangle of sides a and b with respect to an axis lying in the side b.

Solution. Since the area may be considered as made up of thin vertical strips, the moment of inertia of the whole area is obtained from the moment of inertia of one of its elements by substituting the whole area for the area of the element. The result is, therefore,

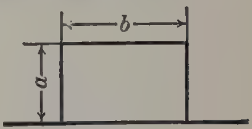

FIG. 231

$$I = \tfrac{1}{3} Aa^2.$$

5. Find the moment of inertia of a rectangular area of sides b and d with respect to an axis through its center of gravity parallel to the side of length b.

Solution. From the parallel-axis theorem and the result of Example 4, the moment of inertia is

$$I = \tfrac{1}{12} Ad^2 = \tfrac{1}{12} bd^3.$$

6. Determine the moment of inertia of the area of an ellipse with respect to one of its principal axes.

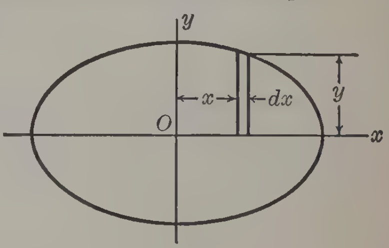

FIG. 232

Solution. The moment of inertia of the element of area with respect to the x axis is $\frac{1}{3}\,dAy^2$, where dA is its area and y is its length. From the equation of the ellipse,

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Also $\qquad dA = y\,dx.$

Therefore $\qquad I = \dfrac{4}{3}\dfrac{b^3}{a^3}\displaystyle\int_0^a (a^2 - x^2)^{\frac{3}{2}}\,dx = \frac{\pi}{4}\,ab^3 = \frac{Ab^2}{4},$

where $\pi ab = A$, the area of the ellipse.

The moment of inertia with respect to the y axis is

$$I = \frac{Aa^2}{4}.$$

7. Find the moment of inertia of an ellipse with respect to a polar axis passing through its center.

Solution. From Example 6 and the theorem of § 116,

$$I = \frac{A}{4}\,(a^2 + b^2).$$

117. The moment of inertia of a plane area with respect to any axis in its plane passing through a point. Let it be required to find the moment of inertia of the plane area shown in Fig. 233 with respect to the axis OP, making an angle θ with the x axis.

Making use of (2), § 114, and noting that the direction cosine $\gamma = 0$, the required moment of inertia is

$$I_{OP} = A\alpha^2 + B\beta^2 - 2\,F\alpha\beta, \quad (1)$$

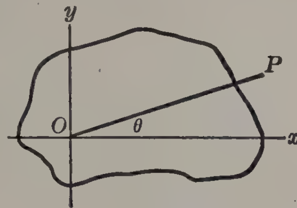

FIG. 233

where A and B are the moments of inertia of the area with respect to the x and y axes, F is the product of inertia with respect to these axes,

$$\alpha = \cos\theta, \quad \text{and} \quad \beta = \cos(90° - \theta) = \sin\theta.$$

Equation (1) may therefore be written

$$I_{OP} = A\cos^2\theta + B\sin^2\theta - 2\,F\sin\theta\cos\theta. \qquad (2)$$

EXAMPLE

Find the moment of inertia of an ellipse of semiaxes a and b with respect to a diameter which makes an angle of 30° with the x axis.

Solution. From symmetry the product of inertia is zero, and therefore

$$I = \frac{Ab^2}{4}\cos^2 30° + \frac{Aa^2}{4}\sin^2 30° = \frac{A}{16}(a^2 + 3\,b^2).$$

118. Principal axes. Let A be the moment of inertia of the plane area, Fig. 234, with respect to the x axis and B the moment of inertia with respect to the y axis. Also let F be the product of inertia of the plane area with respect to the x and y axes.

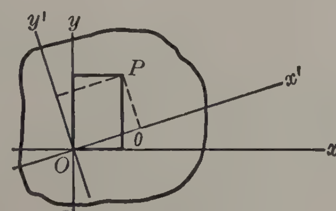

FIG. 234

The moment of inertia of the plane area with respect to a new axis ox' is, by (2), § 117,

$$I_{x'} = A\cos^2\theta + B\sin^2\theta - 2\,F\sin\theta\cos\theta.$$

The value of $I_{x'}$ changes as θ changes and has a maximum and a minimum value which may be determined by the usual method.

Placing the first derivative equal to zero gives

$$\frac{dI_{x'}}{d\theta} = -2\,A\sin\theta\cos\theta + 2\,B\sin\theta\cos\theta - 2\,F\cos^2\theta + 2\,F\sin^2\theta$$

$$= (B - A)\sin 2\,\theta - 2\,F\cos 2\,\theta = 0,$$

from which
$$\tan 2\,\theta = \frac{2\,F}{B - A}. \qquad (1)$$

Therefore $$\sin 2\theta = \frac{2F}{\pm\sqrt{(B-A)^2+4F^2}},$$

and $$\cos 2\theta = \frac{B-A}{\pm\sqrt{(B-A)^2+4F^2}}. \tag{2}$$

For any given values of A, B, and F, (1) gives an infinite number of values of 2θ, of which any two differing by 180° are sufficient to determine two values of θ differing by 90°, and thus to determine the position of the axes of maximum and minimum moments of inertia. The maximum and minimum axes are distinguished by means of the second derivative. The second derivative,

$$\begin{aligned}\frac{d^2I_{x'}}{d\theta^2} &= 2(B-A)\cos 2\theta + 4F\sin 2\theta \\ &= \frac{2(B-A)^2+8F^2}{\pm\sqrt{(B-A)^2+4F^2}},\end{aligned} \tag{3}$$

if the values of $\sin 2\theta$ and $\cos 2\theta$ are substituted from (2). Thus the sign of the second derivative depends on the sign of the radical.

The product of inertia of the plane area with respect to the new axes x' and y' is

$$\begin{aligned}F' &= \Sigma(da \cdot x'y') \\ &= \Sigma[da(x\cos\theta + y\sin\theta)(-x\sin\theta + y\cos\theta)],\end{aligned}$$

from analytic geometry.

Hence $$F' = \frac{A-B}{2}\sin 2\theta + F\cos 2\theta, \tag{4}$$

since $A = \Sigma(y^2\,da)$, $B = \Sigma(x^2\,da)$, and $F = \Sigma(xy\,da)$.

If the product of inertia F' vanishes,

$$\frac{A-B}{2}\sin 2\theta + F\cos 2\theta = 0,$$

or $$\tan 2\theta = \frac{2F}{B-A}. \tag{5}$$

But (5), which expresses the condition that the product of inertia vanishes, is the same as (1), which expresses the condition under which the moment of inertia has its maximum value with respect to the x' axis and its minimum value with respect to the y' axis. Hence, by the definition of principal axes (§ 114), it follows that the principal axes are also the axes of maximum and minimum moments of inertia.

PROBLEMS

1. Find the moment of inertia and radius of gyration of a triangular area with respect to its base b, its altitude being h. *Ans.* $\frac{Ah^2}{6}, \frac{h}{\sqrt{6}}$.

2. Determine the moment of inertia of the triangle of Problem 1 with respect to an axis parallel to the base through its center of gravity. *Ans.* $\frac{Ah^2}{18}$.

3. Find the moment of inertia of a rectangle of sides a and b with respect to a diagonal. *Ans.* $\frac{A}{6}\frac{a^2b^2}{a^2+b^2}$.

4. Show that the moment of inertia of a square is the same with respect to all axes in its plane which pass through its center.

5. Find the product of inertia of a rectangular area of sides a and b with respect to axes coincident with its sides. *Ans.* $\frac{Aab}{4}$.

6. Find the product of inertia of the area A of a right triangle of sides a and b with respect to these sides as axes. *Ans.* $\frac{Aab}{12}$.

7. Find the product of inertia of the area A of the triangle of Problem 6 with respect to parallel axes passing through the center of gravity. *Ans.* $\frac{Aab}{36}$.

8. Find the angles which the principal axes of the triangle of Problem 6 make with the side of length a. *Ans.* $\tan 2\,\theta = \frac{ab}{a^2-b^2}$.

9. Find the angles which the principal axes of the rectangle of Problem 5 make with the side of length a. *Ans.* $\tan 2\,\theta = \frac{3\,ab}{2(a^2-b^2)}$.

119. Moment of inertia of composite bodies. The moment of inertia of a body composed of several parts is equal to the sum of the moments of inertia of the separate parts. Likewise the moment of inertia of a plane area which can be divided into simple areas is equal to the sum of the moments of inertia of the simple areas.

EXAMPLE

Find the moment of inertia of the area of an annulus having an outer radius r_1 and an inner radius r_2 with respect to the polar axis through the center.

Solution. The moment of inertia of the annulus is equal to the moment of inertia of the large circular area minus the moment of inertia of the small circular area. Hence $I = \frac{1}{2}\,\pi r_1^4 - \frac{1}{2}\,\pi r_2^4 = \frac{1}{2}\,A(r_1^2 + r_2^2)$.

This result is valid for a hollow right cylinder having the same cross section as the annulus, if the mass of the cylinder is substituted for the area A of the annulus, since the cylinder may be divided into annular elements.

The principle here used in obtaining the moment of inertia of the cylinder is valid for all solid bodies which have the same cross section everywhere perpendicular to the axis with respect to which the moment of inertia is required.

PROBLEMS

1. Find the moment of inertia of a plane normal section of a box girder consisting of four 2×12 inch planks, with respect to the axis a–a. *Ans.* 2944 in.^{4}

2. Find the moment of inertia of the section of the girder of Problem 1 with respect to an axis b–b parallel to the axis a–a and distant 10 in. from it. *Ans.* 12,544 in.^{4}

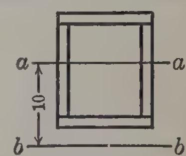

FIG. 235

3. Find the moment of inertia of the section area of a punch frame, Fig. 236, with respect to the axis G–G through the center of gravity. *Ans.* $I = 4411.8$ in.^{4}

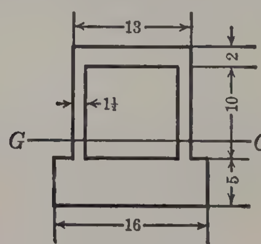

FIG. 236

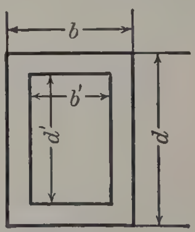

FIG. 237

4. Show that the moment of inertia of the hollow rectangle of Fig. 237 with respect to the axis through the center of gravity parallel to the side of width b is

$$I = \frac{bd^3 - b_1 d_1^{\,3}}{12}.$$

5. Find the product of inertia F of the angle section shown in Fig. 238 with respect to axes parallel to the sides through the center of gravity G.

Ans. $\bar{x} = \bar{y} = 1.87$ in., $F = \pm\, 52.27$ in.^{4}

6. Compute the moment of inertia of the angle section shown in Fig. 238 with respect to the centroidal axes 1–1 and 2–2. *Ans.* 89.0 in.^{4}

7. Determine the greatest and least radii of gyration of the angle section shown in Fig. 238.

Ans. $\theta = 45°$ and $135°$,
$k_1 = 1.56$ in., $k_2 = 3.07$ in.

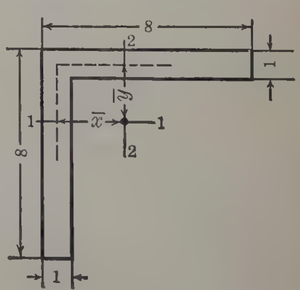

FIG. 238

8. Find the moments of inertia and radii of gyration of the structural I-beam section shown in Fig. 239 with respect to axes 1–1 and 2–2.

Ans. $I_{1-1} = 2087$ in.^{4}, $I_{2-2} = 43$ in.^{4},
$k_{1-1} = 9.46$ in., $k_{2-2} = 1.36$ in.

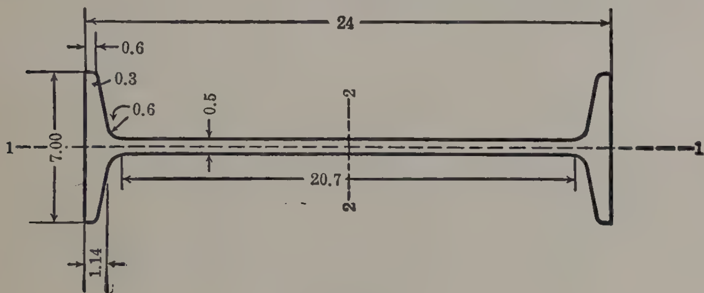

FIG. 239

9. Find the moment of inertia of the Z-bar section shown in Fig. 240 with respect to the axes 1–1 and 2–2.

Ans. 42.15 in.^{4}, 15.44 in.^{4}

10. Find the product of inertia of the Z-bar section, Fig. 240, with respect to the axes 1–1 and 2–2. *Ans.* 18.95 in.^{4}

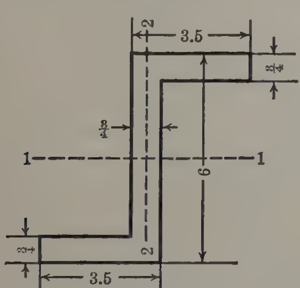

FIG. 240

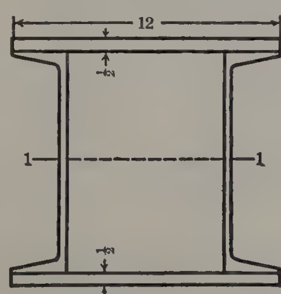

FIG. 241

11. Determine the least radius of gyration of the section of Fig. 240.

Ans. $\theta = 62° \, 36'$, $I = 5.61$ in.^{4}, $k = 0.81$ in.

12. Find the moment of inertia of the 10-inch, 20-pound channel column section shown in Fig. 241 with respect to the axis 1–1. The moment of inertia of the channel section with respect to the axis 1–1 is 78.5 in.^{4} *Ans.* 488 in.^{4}

120. Equimomental bodies. When the moments of inertia of each of two bodies about an axis are equal and this holds for every axis, the two bodies are said to be equimomental.

Two bodies are equimomental when they have the same mass, the same center of gravity, the same principal axes at the center of gravity, and the same moments of inertia about the principal axes. It is customary to distribute the total mass of a body among a number of fictitious points, called equimomental points, which satisfy these conditions. Thus it can be shown that if one third of the area (or mass) of a triangle is concentrated at the mid-point of each side, these points satisfy the above conditions. (Routh's "Rigid Dynamics," Vol. I, Art. 35.) A knowledge of the equimomental points for an actual body is of great service in finding the moments of inertia and products of inertia with respect to axes where the integration is difficult.

121. Partial list of equimomental systems. 1. *Triangular area.* Three particles each equal to one third of the area (or mass) placed at the mid-point of each side.

2. *Parallelogram (area).* Four particles each one sixth of the area (or mass) of the parallelogram placed at the mid-point of each side, and a fifth particle one third of the area (or mass) placed at the center of gravity.

3. *Elliptic area.* Four particles each equal to one fourth of the area (or mass) placed at the mid-points of four oblique lines joining the extremities of the major and minor axes.

4. *Ellipsoid (volume).* One tenth of the mass at each of the six extremities of the principal axes and two fifths at the center.

5. *Tetrahedron (volume).* One twentieth of the mass placed at each of the four vertices and four fifths of the mass at the center of gravity.

PROBLEMS ON EQUIMOMENTAL POINTS

1. Find the moment of inertia of a triangular area with respect to its base b, its altitude being h. Compare result with Problem 1, p. 171.

2. Find the product of inertia of a rectangular area of sides a and b with respect to axes coincident with its sides. *Ans.* $Aab/4$.

3. Find the moment of inertia of a rectangle of sides a and b with respect to a diagonal. Compare result with Problem 3, p. 171.

4. Find the moment of inertia of a solid sphere with respect to an axis tangent to its surface. *Ans.* $\frac{7}{5} Mr^2$.

5. Find the moment of inertia of an elliptic area with respect to an axis tangent to it at the end of the minor axis. *Ans.* $\frac{5}{4} Ab^2$.

6. Find the product of inertia of an elliptic area with respect to two intersecting sides of the circumscribed rectangle. *Ans.* Aab.

KINEMATICS

CHAPTER XIII

PLANE MOTION

122. Introduction. Kinematics is that part of mechanics which is concerned with the motion of bodies, without reference to the causes producing or influencing the motion. The position of a particle can be specified only in a relative sense, that is, with reference to other particles. The position of a particle is usually specified with respect to a system of axes fixed relatively to the earth. Any change of position of a particle constitutes motion of the particle. An important part of the concept of motion is the element of time. The segment of the straight line joining any two positions of a particle is called the *displacement* of the particle. Displacement has both magnitude and direction and is therefore a vector. The *velocity* of a particle is its time rate of change of displacement. The *acceleration* of a particle is its time rate of change of velocity. These changes may be changes in magnitude or direction or both. The motion of a particle is known when its position on its path, its velocity, and its acceleration are determined at every instant of time.

The simplest case of motion, called uniform rectilinear motion, is that in which a particle moves over equal distances along a straight line in any arbitrary equal intervals of time. The distance passed over in one second is called the velocity of the particle.

123. Velocity. Let a point move upon a plane continuous curve C, and let A and B be the positions of the point at times t and $t + \Delta t$. The displacement of the point at the end of the time interval Δt is the line segment or chord AB. Lay off on AB a vector AR having a magnitude equal to the ratio

$$\frac{\text{chord } AB}{\Delta t}.$$

The vector AR is called the *mean velocity* of the particle on the curve C between the positions A and B. The limiting vector AV tangent to the curve at A, which the vector AR approaches as Δt approaches zero, is called the velocity of the moving point on the curve C at the point A. If the arc AB be designated by Δs, where s is measured from some point H on the curve, then the ratio

$$\frac{\text{chord } AB}{\Delta s}$$

approaches unity as Δt approaches zero.

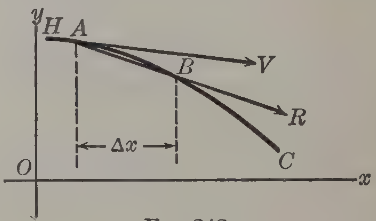

FIG. 242

Hence the magnitude of the velocity is the

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt},$$

and its direction coincides with the tangent at A to the curve C.

From the definition of velocity it is clear that velocity is a vector which requires for complete specification its magnitude, direction, and position. The magnitude of the velocity vector is called *speed*.

It is convenient to specify the velocity by its rectangular components. Let (x, y), $(x + \Delta x, y + \Delta y)$ be the coördinates of the points A and B respectively. The mean component velocities along the axes are

$$\frac{\Delta x}{\Delta t} \quad \text{and} \quad \frac{\Delta y}{\Delta t},$$

which approach the limiting values

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dy}{dt}$$

respectively as Δt approaches zero. The magnitude of the total velocity at the point A is

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$

and its direction is given by

$$\tan \phi = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

124. Acceleration. Let AG and BD be the velocity vectors of a point moving on the curve C at the points $A(x, y)$ and $B(x + \Delta x,\ y + \Delta y)$ at times t and $(t + \Delta t)$ respectively. Let the velocity vector BD be removed parallel to itself to a position AE. It is evident that the velocity vector GE or AF is the vector difference of the two vectors BD and AG, or the change in the velocity during the time interval Δt. In other words, the velocity vector GE or AF must be added vectorially to the first velocity AG to produce the second velocity BD.

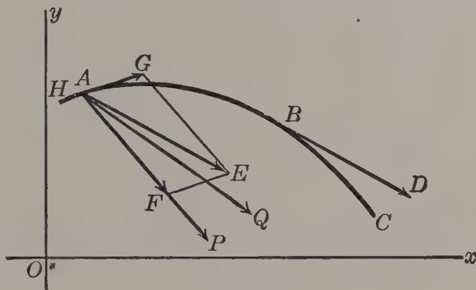

FIG. 243

Upon AF lay off the vector AP having a magnitude $\dfrac{AF}{\Delta t}$. The vector AP is called the mean acceleration of the moving point on the curve C during the time Δt. When Δt approaches zero, the point B approaches A, and the mean acceleration AP approaches a limiting acceleration AQ. The limiting acceleration AQ is called the acceleration of the moving point at the position A.

The magnitude and direction of the acceleration vector AQ may be expressed most simply in terms of its rectangular components. It was shown in § 123 that the velocity $\dfrac{ds}{dt}$ at a point (x, y) could be represented by its rectangular components, $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$. Accordingly the x component of the velocity vector AG at the time t is represented by $\dfrac{dx}{dt}$. Similarly the x component of the velocity vector BD or AE at the time $t + \Delta t$ is represented by $\dfrac{dx}{dt} + \Delta \dfrac{dx}{dt}$, where $\Delta \dfrac{dx}{dt}$ is the increment in $\dfrac{dx}{dt}$ during the time Δt. Since the velocity vectors AG, AE, and GE form a triangle, the x component or projection of the velocity vector GE is the difference of the x components or projections of the velocity vectors AE and AG. Hence the x component of the velocity vector GE is $\Delta \dfrac{dx}{dt}$. Since the velocity vector GE represents the change in velocity of the moving point during the

time Δt, the x component of the mean change in velocity or the x component of the mean acceleration is

$$\frac{\Delta\left(\frac{dx}{dt}\right)}{\Delta t}.$$

As Δt approaches zero, the x component of the mean acceleration approaches the x component of the acceleration AQ and its magnitude approaches

$$\frac{d^2x}{dt^2}.$$

Similarly the magnitude of the y component of the acceleration vector AQ is $\frac{d^2y}{dt^2}$.

The magnitude of the total acceleration vector AQ at the point A on the curve C is

$$a = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2},$$

and its direction is given by

$$\tan\theta = \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}.$$

PROBLEMS

1. A locomotive maintains a constant speed of 60 mi. per hour. Find the speed in feet per second. *Ans.* 88 ft. per second.

2. An aviator flying at a constant speed of 100 ft. per second passes over a distance of 100 mi. Find the time required.
Ans. 1 hr. 28 min.

3. Find the speed of the city of Minneapolis (lat. 45° N.) due to the rotation of the earth on its axis. Assume $R = 4000$ mi. What is the system of reference? Is it in motion? *Ans.* 740.4 mi. per hour.

4. A point moves along the x axis with a constant acceleration k. The initial velocity is v_0 at the origin. Find the velocity and distance traversed after a time interval t. *Ans.* $v = v_0 + kt$, $s = v_0t + \frac{k}{2}t^2$.

5. A point moves on a straight line with constant acceleration, the initial velocity being v_0 and the final velocity v. Show that the distance traversed is the product of the mean velocity $\frac{v_0 + v}{2}$ by the time t, the distance being measured from the point where $t = 0$.

6. If the point in the preceding problem starts from rest, show that the distance it traverses is equal to one half of the final velocity multiplied by the time.

7. A particle having an acceleration of $a_1 + a_2$, where $a_1 > a_2$, starts from rest and traverses a certain distance along a straight line in time t_1. If the time required to traverse the same distance from rest is t_2 when the acceleration is $a_1 - a_2$, show that

$$\frac{t_2}{t_1} = \sqrt{\frac{a_1 + a_2}{a_1 - a_2}}.$$

8. An accelerometer attached to an automobile is observed to read 1 ft. per second per second while the automobile travels a distance of 300 ft. in a straight line. Find the time required if the automobile starts from rest. *Ans.* 24.5 sec.

9. The x and y components of the acceleration of a point are -12 ft. per second per second and 5 ft. per second per second respectively. Find the magnitude and direction of the acceleration.
Ans. 13 ft. per second per second, $\theta = \tan^{-1}(-\frac{5}{12})$.

10. A point moves away from the origin with an acceleration of 20 ft. per second per second along the line $x = \sqrt{3}\, y$. Find the x and y components of the acceleration. Compare the sum of the projections of these components on the line $y = x$ with the projection of the original acceleration on the line $y = x$.
Ans. $\frac{d^2x}{dt^2} = 17.3$ ft. per sec. per sec.; $\frac{d^2y}{dt^2} = 10.0$ ft. per sec. per sec.

11. A point describes the parabola $y^2 = 2\,px$ with uniform speed v; find the x and y components of its speed. *Ans.* $\frac{vy}{\sqrt{p^2 + y^2}}, \frac{vp}{\sqrt{p^2 + y^2}}$.

12. A point on a straight line moves toward a fixed point on the line with a velocity which varies as the third power of the distance x between the fixed point and the moving point. Show that the acceleration of the point varies as the fifth power of x.

13. The space passed over by a particle moving on a straight line in time t is given by $s = r \sin(\omega t + \theta)$, where r, ω, and θ are constants. Determine the velocity and acceleration of the particle at any time t.
Ans. $v = r\omega \cos(\omega t + \theta)$, $a = -r\omega^2 \sin(\omega t + \theta) = -\omega^2 s$.

14. Show that the motion of the particle in Problem 13 is oscillatory with a period of $\frac{2\pi}{\omega}$ and a frequency of $\frac{\omega}{2\pi}$.

15. A point moves upon a straight line with constant acceleration. At times $t = 0$ sec., 1 sec., and 2 sec. the point has traversed distances of 3 ft., 7 ft., and 15 ft. respectively. Find the total distance moved in 8 sec. *Ans.* 147 ft.

125. Angular velocity and angular acceleration of a particle. If a particle moves in a plane, the radius vector drawn from an arbitrary pole to the particle describes an angle whose magnitude measured from a fixed line in the plane varies in general with the time. The time rate of change of the angle is called the *angular velocity* of the particle, and the time rate of change of the angular velocity is called the *angular acceleration* of the particle. Angular velocity and angular acceleration are usually designated by ω and α respectively. Both the angular velocity and the angular acceleration of a particle will in general be different for different poles.

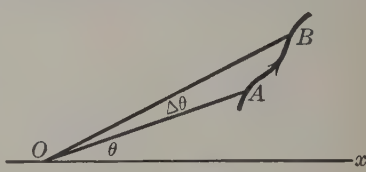

FIG. 244

Let a particle move upon a plane curve, and let A and B represent the positions of the particle at the times t and $t + \Delta t$ respectively. The angular displacement $\Delta\theta$ of the particle referred to any fixed pole O in the plane is the angle between the radius vectors OA and OB. The mean angular velocity of the particle referred to the pole O during the time Δt is

$$\frac{\Delta\theta}{\Delta t}.$$

The *angular velocity* of the particle at A is

$$\omega = \lim_{t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

The angular velocities of the particle at A and B are

$$\frac{d\theta}{dt} \quad \text{and} \quad \frac{d\theta}{dt} + \Delta\frac{d\theta}{dt}$$

respectively. The change in angular velocity in the time interval Δt is therefore $\Delta\left(\frac{d\theta}{dt}\right)$. The mean angular acceleration of the particle during the time Δt is

$$\frac{\Delta\dfrac{d\theta}{dt}}{\Delta t}.$$

The *angular acceleration* of the particle at A is

$$\alpha = \lim_{\Delta t \to 0}\left[\frac{\Delta\dfrac{d\theta}{dt}}{\Delta t}\right] = \frac{d^2\theta}{dt^2}.$$

PROBLEMS

1. A flywheel rotates with a constant angular acceleration α. The initial angular velocity is ω_0 and the initial angle is zero. Find the angular velocity and the angular displacement of any point on the flywheel after a time t.

Ans. $\omega = \omega_0 + \alpha t,\ \theta = \omega_0 t + \frac{\alpha}{2}t^2$.

2. A particle describes a circle. Show that its angular velocity about a fixed point on the circumference is at every instant equal to one half of its angular velocity about the center.

3. Two thin disks are rigidly attached to a shaft so that they are perpendicular to it at points 8 ft. apart. The shaft is rotating at 300 R. P. M. at the instant a bullet moving parallel to the shaft pierces the first disk. The angular acceleration of the shaft is 60 R. P. M. per second. If the angle of separation of the holes in the disks is 30°, find the speed of the bullet. *Ans.* 480 ft. per second.

4. A particle initially at rest at the origin moves along the positive x axis with constant acceleration a. Find the angular velocity and acceleration of the line passing through the particle and the point $(0, b)$.

Ans. $\frac{d\theta}{dt} = \frac{4\,abt}{a^2t^4 + 4\,b^2},\ \frac{d^2\theta}{dt^2} = \frac{16\,ab^3 - 12\,a^3bt^4}{(a^2t^4 + 4\,b^2)^2}$.

126. Radial and transverse velocity (polar coördinates). Let (r, θ) and (x, y) be the polar and rectangular coördinates, respectively, of a particle describing a plane curve. The component of the total velocity of the particle along the radius vector is called the radial velocity of the particle. The component of the total velocity of the particle perpendicular to the radius vector is called the transverse velocity of the particle.

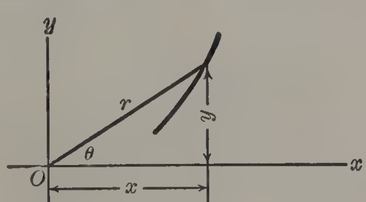

FIG. 245

Since the sum of the projections of the rectangular components of any vector upon any line is equal to the projection of the vector upon that line, the radial velocity of the particle is the sum of the projections of the x and y components of the velocity upon the radius vector. Similarly, the transverse velocity of the particle is the sum of the projections of the x and y components of the velocity upon a line perpendicular to the radius vector. The x and y components of the velocity may be obtained from the relations $x = r \cos \theta$ and $y = r \sin \theta$ by differentiation.

Thus
$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}, \tag{1}$$
and
$$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}. \tag{2}$$

The sum of the projections of the x and y components upon the radius vector is
$$v_r = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta, \tag{3}$$
and on a perpendicular to the radius vector is
$$v_{tr} = -\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta. \tag{4}$$

Substituting the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ from (1) and (2) in the expressions (3) and (4), the radial velocity is
$$v_r = \frac{dr}{dt}, \tag{5}$$
and the transverse velocity is
$$v_{tr} = r\frac{d\theta}{dt} = r\omega. \tag{6}$$

Special case. When the point moves in a circle having its center at the pole, the radial velocity becomes zero and the transverse velocity remains $r\omega$, where r is the radius of the circle.

127. Radial and transverse acceleration (polar coördinates). The radial and transverse components of acceleration of a point moving in a curve may be obtained by projecting the rectangular components of acceleration of the point on the radius vector to the point, and on a perpendicular to the radius vector. The rectangular components of velocity of the point are, from equations (1) and (2), § 126,
$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}, \tag{1}$$
and
$$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}. \tag{2}$$

Differentiating (1) and (2),
$$\frac{d^2x}{dt^2} = \frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}, \tag{3}$$
$$\frac{d^2y}{dt^2} = \frac{d^2r}{dt^2}\sin\theta + 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}. \tag{4}$$

The sum of the projections of the rectangular components of acceleration of the point on the radius vector is

$$\frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta, \tag{5}$$

and on a perpendicular to the radius vector is

$$-\frac{d^2x}{dt^2}\sin\theta + \frac{d^2y}{dt^2}\cos\theta. \tag{6}$$

Substituting the values of $\dfrac{d^2x}{dt^2}$ and $\dfrac{d^2y}{dt^2}$ from (3) and (4) in expressions (5) and (6), the radial acceleration is

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2, \tag{7}$$

and the transverse acceleration is

$$a_{tr} = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}. \tag{8}$$

Special case. When the point moves in a circle having its center at the pole, r is constant and the equations become

$$\text{Radial acceleration} = -r\left(\frac{d\theta}{dt}\right)^2 = -r\omega^2, \tag{9}$$

and $$\text{Transverse or tangential acceleration} = r\frac{d^2\theta}{dt^2} = r\alpha. \tag{10}$$

It is also useful to obtain the *rectangular* components of acceleration of a point moving in a circle. Projecting the accelerations just obtained on the x and y axes respectively,

$$a_x = -r\left(\frac{d\theta}{dt}\right)^2\cos\theta - r\frac{d^2\theta}{dt^2}\sin\theta, \tag{11}$$

and $$a_y = -r\left(\frac{d\theta}{dt}\right)^2\sin\theta + r\frac{d^2\theta}{dt^2}\cos\theta. \tag{12}$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, (11) and (12) may also be written

$$a_x = -x\left(\frac{d\theta}{dt}\right)^2 - y\frac{d^2\theta}{dt^2} = -x\omega^2 - y\alpha, \tag{13}$$

$$a_y = -y\left(\frac{d\theta}{dt}\right)^2 + x\frac{d^2\theta}{dt^2} = -y\omega^2 + x\alpha. \tag{14}$$

EXAMPLE

One end A of a rod traverses a crank circle of radius r_1 with constant angular velocity referred to the center of the circle O_1. The rod slides through a fixed point O in the plane of the circle at a distance l from the center of the circle. The angles made by the rod and the crank with the polar axis are θ and θ_1 respectively. Find the radial and transverse velocity and acceleration of the end A referred to the fixed point O as the pole.

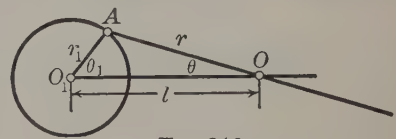

FIG. 246

Solution. Let r be the length of the line joining O to A. Then, from the Law of Cosines,

$$r^2 = r_1^2 + l^2 - 2\,r_1 l \cos\theta_1. \tag{1}$$

Differentiating,
$$2\,r\frac{dr}{dt} = 2\,r_1 l \sin\theta_1 \frac{d\theta_1}{dt},$$

from which the radial velocity is

$$\frac{dr}{dt} = \frac{r_1 l \sin\theta_1 \dfrac{d\theta_1}{dt}}{\sqrt{r_1^2 + l^2 - 2\,r_1 l\cos\theta_1}}. \tag{2}$$

From the figure,
$$\tan\theta = \frac{r_1\sin\theta_1}{l - r_1\cos\theta_1}, \tag{3}$$

and, by differentiation,

$$\sec^2\theta\frac{d\theta}{dt} = \frac{r_1\dfrac{d\theta_1}{dt}(l\cos\theta_1 - r_1)}{(l - r_1\cos\theta_1)^2}. \tag{4}$$

Solving for $\dfrac{d\theta}{dt}$ and replacing $\sec^2\theta$ by its value from (3) gives

$$\frac{d\theta}{dt} = \frac{r_1(l\cos\theta_1 - r_1)\dfrac{d\theta_1}{dt}}{l^2 + r_1^2 - 2\,lr_1\cos\theta_1}. \tag{5}$$

Combining this equation with (1), the required transverse velocity is

$$r\frac{d\theta}{dt} = \frac{r_1(l\cos\theta_1 - r_1)\dfrac{d\theta_1}{dt}}{\sqrt{l^2 + r_1^2 - 2\,lr_1\cos\theta_1}}. \tag{6}$$

Differentiating (2) and (5), remembering that $\dfrac{d^2\theta_1}{dt^2} = 0$, gives the values of $\dfrac{d^2r}{dt^2}$ and $\dfrac{d^2\theta}{dt^2}$.

From these values and (1), (2), and (5) the radial and transverse accelerations are

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$
$$= \left[\frac{3\,r_1^2 l\cos\theta_1 + l^3\cos\theta_1 - 2\,r_1 l^2\cos^2\theta_1 - r_1 l^2 - r_1^3}{(l^2 + r_1^2 - 2\,r_1 l\cos\theta_1)^{\frac{3}{2}}}\right] r_1\left(\frac{d\theta_1}{dt}\right)^2, \tag{7}$$

and
$$a_{tr} = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = \left[-\frac{l\sin\theta_1}{\sqrt{l^2 + r_1^2 - 2\,r_1 l\cos\theta_1}}\right] r_1\left(\frac{d\theta_1}{dt}\right)^2. \tag{8}$$

PROBLEMS

1. A particle describes the spiral $r = k\theta$. Derive expressions for the radial and transverse accelerations of the particle.

Ans. $a_r = k(\alpha - \theta\omega^2)$, $a_{tr} = k(\theta\alpha + 2\,\omega^2)$.

2. A particle describes the cardioid $r = k(1 + \cos\theta)$. Find its radial and transverse accelerations. *Ans.* $a_r = -k[\omega^2(1 + 2\cos\theta) + \alpha\sin\theta]$,
$a_{tr} = k[-2\,\omega^2 \sin\theta + \alpha(1 + \cos\theta)]$.

3. In the example on page 184 let the pole O lie upon the circumference of the circle. Find the radial and transverse accelerations referred to the pole O. *Ans.* $a_r = -\omega_1{}^2 r_1 \sin\dfrac{\theta_1}{2}$, $a_{tr} = -\omega_1{}^2 r_1 \cos\dfrac{\theta_1}{2}$.

4. A point describes a circle whose center is the pole and whose radius is 4 ft. so that the angular velocity of the point is $\omega = 2\,t^3$. Determine the total acceleration after 2 sec., and the angle between it and the radius vector to the moving point.

Ans. $a = 1028.6$ ft. per second per second, $\tan\theta = \frac{3}{32}$.

5. A particle describes a circle of radius 5 ft. The angular velocity of the particle with respect to a pole on the circumference is constant and equal to 4 radians per second. Find the rectangular components of velocity and acceleration of the particle with respect to axes intersecting at the pole, one of which is a diameter, when the radius vector of the particle makes an angle of 30° with the diameter.

Ans. $v_x = -20\sqrt{3}$ ft. per second,
$v_y = 20$ ft. per second,
$a_x = -160$ ft. per second per second,
$a_y = -160\sqrt{3}$ ft. per second per second.

6. A particle moves in a plane about a fixed pole in the plane so that its radial velocity is constant and equal to 8 ft. per second and its transverse velocity is constant and equal to 4 ft. per second. The particle starts from the point (2, 0) at the time $t = 0$. Find the polar equation of the path of the particle, and the total angle described by the radius vector to the particle in 12 sec. Also find the radial and transverse accelerations of the particle when $t = 12$ sec.

Ans. $r = 2\,e^{2\theta}$,
$\theta = \log_e 7 = 111.5°$,
$a_r = -\frac{8}{49}$ ft. per second per second,
$a_{tr} = \frac{16}{49}$ ft. per second per second.

7. A particle travels along a straight tube with a uniform acceleration a, while the tube rotates with uniform angular velocity ω about an axis perpendicular to the tube and passing through a fixed point of the tube. Find the radial and transverse accelerations of the particle after time t, taking the fixed point as a pole. The particle starts from rest at the pole when t is zero. *Ans.* $a_r = a - \frac{1}{2}\,at^2\omega^2$, $a_{tr} = 2\,at\omega$.

128. Tangential and normal velocity and acceleration. There are two directions at any point upon a plane curve which are independent of the axes of reference, namely, the direction of the tangent and the direction of the normal. The projection of the total velocity or acceleration of a moving particle at a point on a plane curve upon the tangent and normal at that point are called the tangential and normal components of the velocity or acceleration of the particle at that point.

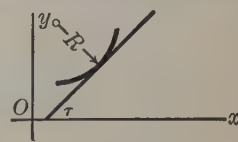

FIG. 247

The velocity of a particle moving on a plane curve is $\frac{ds}{dt}$ (§ 123). The direction of the velocity vector is along the tangent, and therefore the tangential velocity is $\frac{ds}{dt}$ and the normal velocity is zero.

Since the x component of the velocity is the projection of the total velocity on the x axis,

$$\frac{dx}{dt} = \frac{ds}{dt}\cos\tau, \tag{1}$$

where τ is the angle which the tangent to the curve at the point (x, y) makes with the positive x axis. Similarly the y component of the velocity is

$$\frac{dy}{dt} = \frac{ds}{dt}\sin\tau. \tag{2}$$

Differentiating (1) and (2),

$$\frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}\cos\tau - \sin\tau\,\frac{ds}{dt}\frac{d\tau}{dt}, \tag{3}$$

and

$$\frac{d^2y}{dt^2} = \frac{d^2s}{dt^2}\sin\tau + \cos\tau\,\frac{ds}{dt}\frac{d\tau}{dt}. \tag{4}$$

The tangential and normal components of the acceleration may be obtained by projecting the rectangular components of the acceleration on the tangent and normal. The tangential acceleration is, therefore,

$$a_t = \frac{d^2x}{dt^2}\cos\tau + \frac{d^2y}{dt^2}\sin\tau, \tag{5}$$

and the normal acceleration is

$$a_n = \frac{d^2x}{dt^2}\sin\tau - \frac{d^2y}{dt^2}\cos\tau. \tag{6}$$

Substituting the values of $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ from (3) and (4) in (5) and (6),

$$a_t = \frac{d^2s}{dt^2} = \frac{dv}{dt}, \tag{7}$$

and
$$a_n = -\frac{ds}{dt}\frac{d\tau}{dt} = -\frac{v^2}{R}, \tag{8}$$

since, from calculus, the radius of curvature $R = \frac{ds}{d\tau}$ and also $\frac{d\tau}{dt} = \frac{d\tau}{ds}\frac{ds}{dt}$. Also, since the positive direction of the normal is taken outward or from the concave to the convex side of the curve, the negative sign of the normal acceleration shows that it is directed toward the center of curvature.

In the special case where the curve is a circle of radius r the tangential acceleration a_t is, from (7),

$$a_t = \frac{dv}{dt} = \frac{d}{dt}\left(r\frac{d\theta}{dt}\right) = r\frac{d^2\theta}{dt^2} = r\alpha, \tag{9}$$

and
$$a_n = -\frac{v^2}{r} = -\frac{\left(r\frac{d\theta}{dt}\right)^2}{r} = -r\omega^2, \tag{10}$$

where ω and α are the angular velocity and acceleration, respectively, of the particle with respect to the center of the circle.

EXAMPLES

1. Find the linear velocity and the normal, tangential, and total acceleration of a point on the rim of a flywheel 6 ft. in diameter. The wheel rotates at 20 R.P.M., and the speed of the wheel is being increased at the uniform rate of one revolution per second per second.

Solution. The linear velocity of the point is $r\omega$. The normal component of the acceleration is
$$a_n = -\omega^2 r.$$
The tangential acceleration is
$$a_t = r\alpha,$$
and the total acceleration is
$$a = \sqrt{(-r\omega^2)^2 + (r\alpha)^2}.$$

From the problem, $\omega = \frac{2\pi}{3}$ and $\alpha = 2\pi$.

Hence the linear velocity is $r\omega = 3\left(\frac{2\pi}{3}\right) = 6.28$ ft. per second.

Also
$$a_n = 3\left(\frac{4\pi^2}{9}\right) = 13.16 \text{ ft. per second per second,}$$
$$a_t = 3(2\pi) = 18.85 \text{ ft. per second per second,}$$
and the total acceleration is
$$a = \sqrt{(13.16)^2 + (18.85)^2} = 22.99 \text{ ft. per second per second.}$$

The angle between the total acceleration vector and the tangent to the rim of the flywheel at the point is given by

$$\tan \phi = \frac{13.16}{18.85} = 0.698.$$

2. A car travels around an elliptic path whose equation is

$$\frac{x^2}{(100)^2} + \frac{y^2}{(50)^2} = 1$$

with a constant speed of 30 ft. per second. Find the normal and tangential acceleration of the car when it passes through the extremities of the major and minor axes.

Solution. The radii of curvature of an ellipse at the extremities of the major and minor axes are

$$\frac{b^2}{a} = 25 \quad \text{and} \quad \frac{a^2}{b} = 200$$

respectively. Hence at the extremity of the major axis

$$a_n = \frac{v^2}{R} = \frac{(30)^2}{25} = 36 \text{ ft. per second per second,}$$

and at the extremity of the minor axis

$$a_n = \frac{(30)^2}{200} = 4.5 \text{ ft. per second per second.}$$

The tangential acceleration is zero at all points.

PROBLEMS

1. A point describes a circle with uniform speed which is numerically equal to four times its linear acceleration. Find the angular velocity in revolutions per minute if the radius of the circle is 16 ft.

Ans. 2.387 R.P.M.

2. A point describes a circle having a radius of 16 ft. The total acceleration of the point is directed toward the center of the circle, and its magnitude is 64 ft. per second per second. What is the speed?

Ans. 32 ft. per second.

3. A locomotive runs upon a circular track having a radius of $\frac{1}{2}$ mi. If the acceleration along the tangent is constant and equal to $\frac{1}{8}$ ft. per second per second, how long will it take the locomotive to return to its starting point? The initial speed is zero. What will be its speed when it passes the starting point? What is its total acceleration as it passes the starting point?

Ans. $t = 515.2$ sec.,
$v = 64.4$ ft. per second,
$a = 1.57$ ft. per second per second.

4. A point describes a circle of radius 16 ft. The total acceleration vector has a magnitude of 32 ft. per second per second, and it makes an angle of 60° with the radius drawn to the moving point. Find the speed of the particle and the acceleration along the tangent.

Ans. 16 ft. per second, $a = 27.71$ ft. per second per second.

5. A particle starting from the origin travels on the spiral $\rho = 3\,\theta$ with a constant speed of 5 ft. per second. Find the normal and tangential components of its acceleration and the total acceleration when $\theta = 2\,\pi$. Check the total acceleration by finding the resultant of the radial and transverse accelerations.

6. A lazy tongs consisting of bars 1 ft. long is pivoted at a fixed point A as shown in Fig. 248. The joint C is constrained to move along AC. Find the velocity and acceleration of C when $\theta = 60°$ and ω is constant and equal to 2 radians per second.

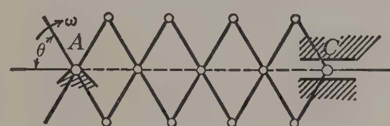

FIG. 248

$$\textit{Ans.}\ \frac{dx}{dt} = -4\sqrt{3} \text{ ft. per second,}$$

$$\frac{d^2x}{dt^2} = -8 \text{ ft. per second per second.}$$

7. Find the speed and acceleration of the end of a string B which is unwound from a cylinder of radius r in such a manner that the point of contact A turns with uniform angular velocity ω with respect to O.

HINT. From the figure

$$x = r\cos\omega t + r\omega t\sin\omega t,$$
$$y = r\sin\omega t - r\omega t\cos\omega t.$$

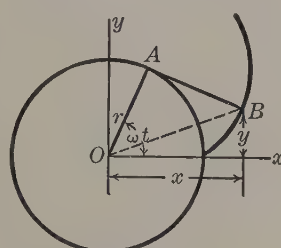

FIG. 249

$$\textit{Ans.}\ v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = r\omega^2 t = \overline{AB}\,\omega,$$

$$a = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} = r\omega^2\sqrt{1 + \omega^2 t^2} = \overline{OB}\omega^2.$$

8. A wheel of radius r rolls at uniform speed along the floor. A point on the rim describes a cycloid whose parametric equations are

$$x = r\theta - r\sin\theta,$$
$$y = r - r\cos\theta.$$

Find the rectangular components of velocity and acceleration of the point.

$$\textit{Ans.}\ \frac{dx}{dt} = r\omega(1 - \cos\theta), \quad \frac{d^2x}{dt^2} = r\omega^2\sin\theta,$$

$$\frac{dy}{dt} = r\omega\sin\theta, \quad \frac{d^2y}{dt^2} = r\omega^2\cos\theta.$$

9. A light stylus attached to a fixed tuning fork making n complete vibrations per second traces a wavy line on the smoked surface of a rotating flywheel at a distance r from its center. The length of k waves is found to be p and the length of the next k waves is found to be q. Find the angular acceleration of the flywheel, assuming that it is constant. *Ans.* $\dfrac{(q-p)n^2}{rk^2}$.

10. A particle moves on the curve $y^2 = 16 - x$ so that $x = t^2$. Find a_x and a_y, a_n and a_t, a_r and a_{tr} at the end of 3 sec. Check by finding the total acceleration in each case.

129. Summary. The results of the previous articles are tabulated for convenience of reference.

FOR A POINT MOVING ALONG ANY PLANE CURVE

	VELOCITY	ACCELERATION	POSITIVE IN THE POSITIVE DIRECTION OF
Along axis of x	$\dfrac{dx}{dt}$	$\dfrac{d^2x}{dt^2}$	x
Along axis of y	$\dfrac{dy}{dt}$	$\dfrac{d^2y}{dt^2}$	y
Along radius vector	$\dfrac{dr}{dt}$	$\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$	r
Transverse to radius vector	$r\dfrac{d\theta}{dt}$	$2\dfrac{dr}{dt}\dfrac{d\theta}{dt} + r\dfrac{d^2\theta}{dt^2}$	θ
Along tangent	$\dfrac{ds}{dt}$	$\dfrac{d^2s}{dt^2}$	s
Along outward normal . .	0	$-\dfrac{v^2}{R}$	R

FOR A POINT MOVING ALONG A CIRCLE

	VELOCITY	ACCELERATION	POSITIVE IN THE POSITIVE DIRECTION OF
Along axis of x	$\dfrac{dx}{dt}$	$\dfrac{d^2x}{dt^2}$	x
Along axis of y	$\dfrac{dy}{dt}$	$\dfrac{d^2y}{dt^2}$	y
Along radius vector or outward normal	0	$-r\left(\dfrac{d\theta}{dt}\right)^2$ or $-\dfrac{v^2}{r}$	θ, r
Transverse to radius vector or along tangent	$\dfrac{ds}{dt}$ or $r\dfrac{d\theta}{dt}$	$r\dfrac{d^2\theta}{dt^2}$ or $\dfrac{d^2s}{dt^2}$	r, s

130. Plane motion of a rigid body. A rigid body has *plane motion* when during the motion each particle of the body remains at a constant distance from a fixed plane, or when every particle of the rigid body lying in a fixed plane remains in that plane. The fixed plane is called the plane of motion.

The exact position of a rigid body having plane motion is known if the positions of two points of the body lying in the plane are known. Therefore in the study of the plane motion of a rigid body, the rigid body may be replaced by a plane coinciding with and sliding upon the fixed plane.

The plane motion of a rigid body may consist of a translation or rotation or both.

131. Translation of a rigid body. A rigid body has a motion of *translation* if during the motion a set of rectangular axes fixed in the body remains parallel to a set of axes fixed in space. The motion of translation of a body is called rectilinear translation or curvilinear translation according as any particle of the body describes a straight line or a curve. The side rod of a locomotive traveling on a straight level track has a motion of curvilinear translation, since each particle of the rod describes equal prolate cycloids. If the wheels are locked and slide on the rails, the motion of the side rod is rectilinear translation. It is evident that in the motion of translation of a rigid body, all the particles have the same velocity at the same instant, and likewise they all have the same acceleration at the same instant. The velocity and acceleration of a rigid body having a motion of translation are the velocity and acceleration of any particle of the body.

132. Rotation of a rigid body. Let ox and oy be fixed rectangular axes, and let ox' and oy' be a set of rotating rectangular axes having the same origin and lying in the same plane as the fixed axes. The angle between the axes ox and ox' will in general change with the time, and the angular velocity of the rotating axes is the time rate of change of that angle. The time rate of change of the angular velocity of the rotating axes is the angular acceleration of the rotating axes. A rigid body attached to the rotating axes is said to have rotation about an axis passing through the origin perpendicular to the plane of motion. It is evident that all particles of the body describe circles whose centers lie on the axis. The angular velocity and angular acceleration of the body

are the angular velocity and angular acceleration of the rotating axes with respect to the fixed axes. The linear velocity of any particle of the body depends upon its distance from the axis of rotation and upon the angular velocity of the body.

133. Any displacement of a body in plane motion is equivalent to a rotation. Let any line segment AB of a rigid body be displaced in a plane to a position $A'B'$. The displacement may be effected by a rotation about an axis perpendicular to the plane. Let O be the point of intersection of the perpendicular bisectors of AA' and BB'. Then since the triangles AOB and $A'OB'$ are equal, the displacement of the line segment from the position AB to the position $A'B'$ can be effected by a rotation about an axis through O perpendicular to the plane. Also, since the line segment is fixed in the body, the displacement of the body can be effected by a rotation about the same axis. An apparent exception occurs when the perpendicular bisectors are parallel. In this case the point of intersection is infinitely distant and the displacement is effected by a translation, that is, a rotation about an infinitely distant center.

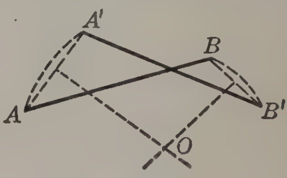

FIG. 250

134. Any displacement of a body in plane motion is equivalent to a rotation about an axis perpendicular to the plane of motion and passing through any point of the body, and a translation of the body. Let AC and $A''C''$ be the initial and final positions of any straight line of the body lying in the plane of motion. Join the initial and final positions of an arbitrary point B of the line. Evidently the line AC can be brought to the final position $A''C''$ by a translation represented by BB'' followed by a rotation about B''. The angular displacement or rotation is independent of the point B, but the translation depends upon the position of B. Since these displacements are independent, they may occur in any order or simultaneously. Also, since the line segment is fixed in the body, the displacement of the body is effected by the same translation and

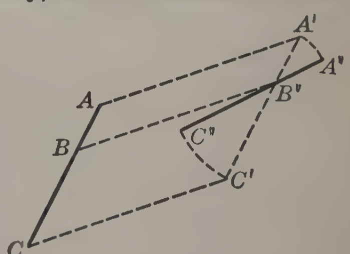

FIG. 251

rotation. The successive positions of the line or body during a translation and a rotation are not in general coincident with the positions of the line or body during the actual motion.

135. Instantaneous axis. The actual plane motion of any body may be considered as being composed of a succession of infinitesimally small displacements. In § 133 it is shown that any displacement of a rigid body having plane motion may be effected by a pure rotation about an axis perpendicular to the plane of motion. Let any line segment AB of the body be displaced to a neighboring position $A'B'$ during the time Δt. Then, in the actual motion, as Δt approaches zero the center of rotation O approaches a limiting position for the initial position of the line AB. This limiting position of the point O is called the instantaneous center. In general, as the line or body moves the instantaneous center changes its position in the body and also with reference to fixed space. The locus of the positions of the point O referred to the body is called the *body centrode*. The locus of the positions of the point O referred to fixed space is called the *space centrode*. A wheel rolling on the ground affords a simple example. The instantaneous center is the point of contact of the wheel on the ground. As the wheel rolls along, the point of contact moves along the ground and successive points of the rim of the wheel become in turn the point of contact. The rim of the wheel is therefore the body centrode, and the track of the wheel is the space centrode. The actual motion of the wheel is effected by rolling the body centrode on the space centrode. In general, the plane motion of any body may be exactly reproduced by rolling the body centrode upon the space centrode.

EXAMPLE

A slender bar moves so that one end is on the x axis and the other on the y axis. Determine the instantaneous axis for any position of the bar, and find the space centrode and the body centrode.

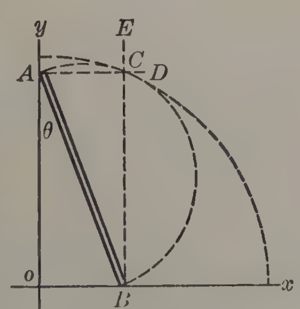

FIG. 252

Solution. Since the extremity A of the bar moves along oy, the instantaneous center must lie in the perpendicular AD. Likewise, since the extremity B moves along ox, the instantaneous center must lie on the perpendicular BE. Hence the point of intersection C is the instantaneous center. Since ACB is a right angle

for every position of the bar, the locus of the instantaneous center C relative to the bar (body centrode) is a circle on AB as a diameter. Moreover, since $oC = AB$ for every position of the bar, the space centrode is a circle of radius AB having its center at the origin. The motion of the bar may be completely reproduced by rolling the circle ACB inside the larger circle. A graphical construction of the two centrodes may easily be made. An experimental verification of the construction by means of a cardboard model is instructive.

PROBLEMS

1. In a steam-engine mechanism the length of the connecting rod and the radius of the crank circle are given. Find the position of the instantaneous center of the connecting rod in any position. Show how to find the instantaneous velocity of any point of the connecting rod. Plot the body and space centrodes.

2. A rod moves with its extremities on two straight lines which intersect. Find the position of the instantaneous center for any position of the rod. Plot the body and space centrodes.

3. A wheel 6 ft. in diameter rolls along a level road with a speed of 10 ft. per second. Show how the instantaneous center may be used to determine the velocity of any point in the wheel. Determine the velocity of four points on the rim of the wheel 90° apart, one of the points being at the instantaneous center.

Ans. 0 ft. per second, $10\sqrt{2}$ ft. per second, $10\sqrt{2}$ ft. per second, 20 ft. per second.

4. Show that the projections of the velocities of any two points of a rigid body upon the line joining them are equal.

5. The bar shown in Fig. 252 is 6 ft. long, and the angle θ is 30°. Find the velocity of A if the velocity of B is 20 ft. per second. Show that the projections of the velocities of A and B on AB are equal.

Ans. $\frac{20}{\sqrt{3}}$ ft. per second.

6. A locomotive whose drive wheels are 6 ft. in diameter moves to the right with a speed of 8 ft. per second, while the wheels slip clockwise at the rate of one revolution per second. Locate the instantaneous center of one of the wheels.

Ans. 1.27 ft. below its center.

7. A string is unwound from a cylinder so that the string remains taut. Locate the instantaneous center of the unwound portion of the string. Show how to draw a tangent to the curve described by the end of the string.

8. The triangular plate ABC is constrained to move in its plane by links CE and BD, which rotate about fixed centers D and E. Locate the instantaneous center of the plate ABC in the position shown in Fig. 253, and determine the linear velocity of the vertex A if the angular velocity ω of BD is 10 R. P. M.

Ans. 18.14 ft. per second.

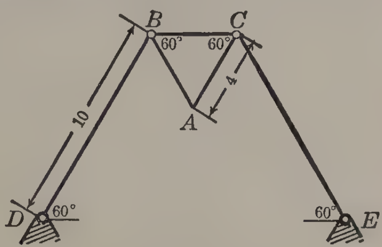

FIG. 253

9. The lengths of the crank and connecting rod of a steam engine are r and l respectively. The connecting rod makes an angle ϕ with the axis of the cylinder, and the crank makes an angle θ with the axis. Show that the velocity of the crosshead is

$$r\frac{d\theta}{dt}\sin\theta\left(1 \pm \frac{r\cos\theta}{\sqrt{l^2 - r^2\sin^2\theta}}\right).$$

136. Composition of motions. Let a particle P (Fig. 254) move along a path fixed to a system of axes $x_r o_r y_r$, which axes are themselves in motion with respect to a set of axes xoy, which are regarded as fixed. The path, velocity, and acceleration of the particle P with respect to the moving axes $x_r o_r y_r$ are called the *relative path, relative velocity,* and *relative acceleration* of the particle. The path, velocity, and acceleration of the particle P referred to the fixed axes xoy are called the *absolute path, absolute velocity,* and *absolute acceleration* of the particle. The displacement, velocity, and acceleration of a point P' fixed relative to the moving axes $x_r o_r y_r$ and through which the particle P is moving are called the *entrained displacement, entrained velocity,* and *entrained acceleration* of the particle P. The entrained displacement, velocity, and acceleration of the particle P may also be defined as the displacement, velocity, and acceleration which the particle would have if it did not move on the relative path while the relative path continued its motion.

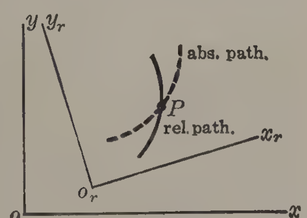

FIG. 254

The general method of combining motions in a plane is shown in Fig. 255, where the moving axes $x_r o_r y_r$ occupy the position (A) with respect to the fixed axes xoy at the time t and the position (B) at the time $(t + \Delta t)$. The particle P moves upon the relative

path M from the initial position P in (A) to the position Q in (B) during the time Δt. During the same time Δt the particle P describes the absolute path N from P to Q. The absolute displacement, the entrained displacement, and the relative displacement are represented by a, e, and r respectively. It is evident from the figure that the absolute displacement is equal to the vector sum of the entrained displacement and the relative displacement.

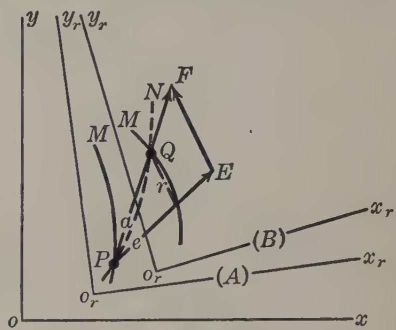

FIG. 255

Lay off vectors PF, PE, and EF equal to $\frac{a}{\Delta t}$, $\frac{e}{\Delta t}$, and $\frac{r}{\Delta t}$ respectively. These vectors represent the average absolute velocity, the average entrained velocity, and the average relative velocity of the particle P. When Δt approaches zero the point Q approaches the point P and the average velocity vectors approach the limiting velocity vectors V_a, V_e, and V_r, Fig. 256. Hence it follows that if a particle moves on a path fixed with reference to moving axes, the absolute velocity of the particle is the vector sum of the velocity of the point which the particle occupies at the instant in question referred to the fixed axes and the relative velocity of the particle with respect to the moving axis, or, in vector form,

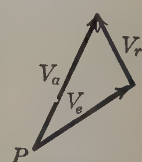

FIG. 256

$$\overline{V_a} = \overline{V_e} + \overline{V_r}.$$

137. Composition of translations. Let a body A move with a velocity of translation v_A with respect to axes xoy, and let a second body B move with a velocity of translation v_B with respect to a set of axes $x_r o_r y_r$ fixed in the body A. Since in a motion of translation all particles of the body have the same velocity, it follows from § 136 that the

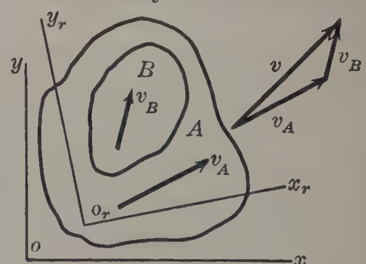

FIG. 257

velocity of translation of the body B with respect to the axes xoy is the vector sum of the entrained velocity v_A and the relative velocity v_B. It is evident that any number of such translations may be combined in a similar manner.

138. Composition of rotations about parallel axes. Let a body A rotate about an axis through o which is perpendicular to the plane of motion containing the axes xoy. Also let ω_A be the angular velocity of the body A about the axis through o. As the body A rotates it carries with it a parallel axis through o_r, about which a second body B rotates with an angular velocity ω_B referred to an axis o_rx_r fixed in the body A lying in the plane of motion. The velocity of any particle P of the body referred to axes xoy is the vector sum of the velocity of the point of body A which instantaneously coincides with the particle P and the relative velocity of the particle P with respect to the axes $x_ro_ry_r$ (§ 136). Thus in Fig. 258 the velocity of the particle P referred to the axes xoy is the vector sum of $\omega_A r_A$ perpendicular to r_A and $\omega_B r_B$ perpendicular to r_B.

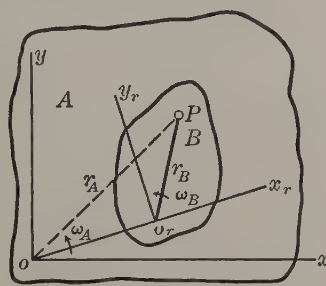

FIG. 258

139. Relative velocity and acceleration (parallel axes). Let the velocity and acceleration of a particle P be given with respect to a set of moving rectangular axes $x_ro_ry_r$ which remain parallel to a set of fixed axes xoy. Such a set of moving axes will be called translating axes. Also let the velocity and acceleration of the translating axes be given with respect to the fixed axes. It is required to find the velocity and acceleration of the particle P with respect to the fixed axes.

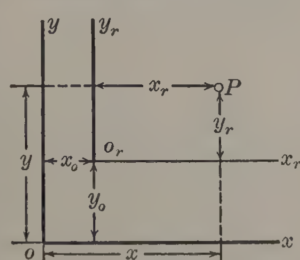

FIG. 259

From Fig. 259 it is clear that

$$x = x_o + x_r \tag{1}$$

and

$$y = y_o + y_r. \tag{2}$$

Differentiating (1) and (2) with respect to the time gives

$$\frac{dx}{dt} = \frac{dx_o}{dt} + \frac{dx_r}{dt}, \tag{3}$$

and

$$\frac{dy}{dt} = \frac{dy_o}{dt} + \frac{dy_r}{dt}. \tag{4}$$

These equations express the components of the absolute velocity, $\frac{dx}{dt}$ and $\frac{dy}{dt}$, in the terms of the components of the relative velocity, $\frac{dx_r}{dt}$ and $\frac{dy_r}{dt}$, and the components of the entrained velocity, $\frac{dx_o}{dt}$ and $\frac{dy_o}{dt}$, of the translating axes. Equations (3) and (4) may be interpreted vectorially as follows:

The absolute velocity of the particle P is the vector sum of the velocity of the particle with respect to the translating axes and the absolute velocity of the translating axes.

Conversely, the relative velocity of the particle P is the vector difference of the absolute velocity of the particle and the absolute velocity of the translating axes.

The last two statements are true whether the moving axes translate or rotate (§ 140).

Differentiating (3) and (4) with respect to the time gives

$$\frac{d^2x}{dt^2} = \frac{d^2x_o}{dt^2} + \frac{d^2x_r}{dt^2} \tag{5}$$

and

$$\frac{d^2y}{dt^2} = \frac{d^2y_o}{dt^2} + \frac{d^2y_r}{dt^2}. \tag{6}$$

It therefore follows that *the absolute acceleration of the particle P is the vector sum of the acceleration of the particle with respect to the translating axes and the absolute acceleration of the translating axes.*

Conversely, the relative acceleration of the particle P is the vector difference of the absolute acceleration of the particle and the absolute acceleration of the translating axes.

The last two statements are true when the moving origin moves on any curve in the plane of motion, but the moving axes must either remain parallel to the fixed axes or make a constant angle with them.

EXAMPLES

1. A train A moves east with a velocity of 30 mi. per hour, and a second train B moves south with a velocity of 40 mi. per hour. Find the relative velocity of the train A with respect to the train B. Also find the relative velocity of the train B with respect to the train A.

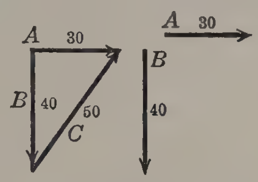

FIG. 260

Solution. The relative velocity of the train A with respect to the train B is the vector difference of the absolute velocities of the trains A and B. Hence the relative velocity of the train A with respect to the train B is represented by the vector C (Fig. 260). The relative velocity of the train B with respect to the train A is the vector C with the sense reversed.

2. A bar AB rotates in a plane about a fixed point o in the plane with an angular velocity $\omega = -4$ and angular acceleration $\alpha = -6$. A bar BC rotates in the same plane about an axis B with an absolute angular velocity $\omega_1 = -6$ and an absolute angular acceleration $\alpha_1 = 8$. Find the x and y components of the absolute acceleration of the point C in the position shown in Fig. 261.

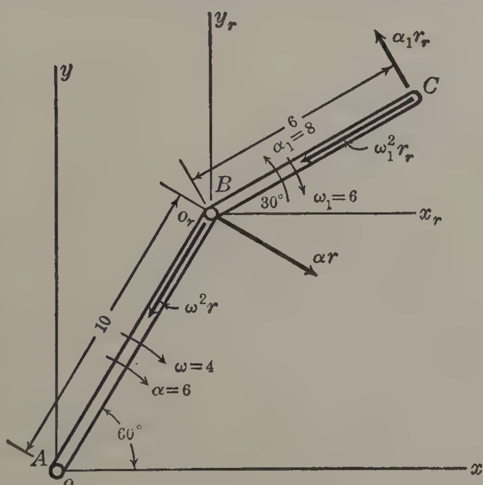

FIG. 261

Solution. Let a system of axes $x_r o_r y_r$ have its origin at B and move parallel to the system of axes xoy. The x component of the absolute acceleration of the point C is equal to the sum of the x component of the acceleration of the point B and the x component of the relative acceleration of the point C with respect to the axes $x_r o_r y_r$. The absolute acceleration of the point B is given by its radial and transverse components,

$$\omega^2 r = 4^2(10) = 160 \text{ ft. per second per second}$$

and

$$\alpha r = 6(10) = 60 \text{ ft. per second per second.}$$

Likewise the relative acceleration of the point C is given by its radial and transverse components,

$$\omega_1{}^2 r_r = 6^2(6) = 216 \text{ ft. per second per second}$$

and

$$\alpha_1 r_r = 8(6) = 48 \text{ ft. per second per second.}$$

The x and y components of the absolute acceleration of the point C are, therefore,

$$\frac{d^2x}{dt^2} = -160 \cos 60° + 60 \cos 30° - 216 \cos 30° - 48 \cos 60°$$
$$= -239.1 \text{ ft. per second per second,}$$

and
$$\frac{d^2y}{dt^2} = -160 \sin 60° - 60 \sin 30° - 216 \sin 30° + 48 \sin 60°$$
$$= -235.0 \text{ ft. per second per second.}$$

3. Find the absolute acceleration of any point of the rim of a wheel which rolls along the ground.

Solution. The absolute acceleration of any point P on the rim is equal to the vector sum of its acceleration referred to parallel axes whose origin o_r coincides with the center of the wheel and the acceleration of the origin o_r. The relative accelerations of the point P are $\omega^2 r$ and αr, as shown in Fig. 262. The acceleration of o_r is αr. Hence the horizontal and vertical components of the absolute acceleration are

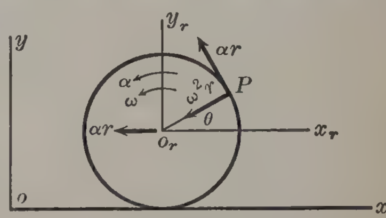

FIG. 262

$$\frac{d^2x}{dt^2} = -\alpha r - \omega^2 r \cos \theta - \alpha r \sin \theta,$$

$$\frac{d^2y}{dt^2} = -\omega^2 r \sin \theta + \alpha r \cos \theta.$$

PROBLEMS

1. A train A running east at 60 mi. per hour is followed by a train B running in the same direction at 40 mi. per hour. Find the relative velocity of train A with respect to train B and the relative velocity of train B with respect to train A. Also find the velocity of train A with reference to an observer walking east on train B at 4 mi. per hour.
Ans. 20 mi. per hour east,
20 mi. per hour west,
16 mi. per hour east.

2. A hailstone falls vertically with a speed of 88 ft. per second. To a man riding in a train at 60 mi. per hour, at what angle does it appear to drop? What is its velocity with respect to the train?
Ans. 45°, 124.4 ft. per second.

3. The position of two particles, A and B, on a line at any time t is given by the equations $x_A = 2t^2 + 3t + 4$ and $x_B = 5t^2 + t + 2$. Determine the relative velocity of A with respect to B when the distance between the particles is a minimum. *Ans.* 0.

4. A man riding on a train traveling 90 ft. per second throws a ball in a horizontal direction perpendicular to the train at a speed of 60 ft. per second. Find the initial absolute velocity of the ball.
Ans. 108.2 ft. per second.

5. The length of the crank of a locomotive is 13 in. and the diameter of the drive wheel is 60 in. Find the absolute velocity of the crank pin at its highest position when the speed of the locomotive is 30 mi. per hour. Also find the relative velocity of the crank pin with respect to the engine frame. *Ans.* 63.1 ft. per second, 19.1 ft. per second.

6. An airplane flies a distance AB of 100 mi. in 1 hr. 15 min. A constant wind blows at right angles to AB with a velocity of 30 mi. per hour. Find the velocity of the airplane in still air.

Ans. 85.44 mi. per hour.

7. Find the time required for an airplane having a speed of 90 mi. per hour in still air to fly from A to B if B is 100 mi. east of A and a southeast wind having a velocity of 30 mi. per hour is encountered. Also find the time required for the airplane to fly from B to A. *Ans.* 90.6 min., 55.2 min.

8. An aviator whose velocity in still air is v mi. per hour flies around a square course whose sides are s mi. A wind having a velocity of u mi. per hour blows constantly at right angles to a side of the square. Find the time required to fly around the course.

$$Ans.\ 2\,s\left(\frac{v+\sqrt{v^2-u^2}}{v^2-u^2}\right).$$

9. The hour-hand and the minute-hand of a clock are together at 12 o'clock. Find the time interval between the successive meetings of the hands. *Ans.* $\frac{12}{11}$ hr.

10. A passenger on a train traveling at 60 mi. per hour observes that it requires 4 sec. for a train 528 ft. long to pass him. Find its velocity. *Ans.* 30 mi. per hour or 150 mi. per hour.

11. Two particles having constant speeds describe concentric circles whose radii are 6 ft. and 9 ft. The radii of the particles coincide at intervals of 3 sec. and 1 sec. according as the particles travel in the same or opposite directions. Find the speed of the particles.

Ans. $4\,\pi$ ft. and $12\,\pi$ ft. per second,
or $6\,\pi$ ft. and $8\,\pi$ ft. per second.

12. A wheel rolls along the ground in a vertical plane. Find the total acceleration of the point of contact and show that it passes through the center of the wheel. *Ans.* $\dfrac{d^2x}{dt^2}=0,\quad \dfrac{d^2y}{dt^2}=\omega^2 r.$

13. A set of axes $x'o'y'$ which remain parallel to a set of fixed axes xoy has an acceleration of 18 ft. per second per second at an angle of 60° to the x axis. A particle has an acceleration of 26 ft. per second per second at an angle of 30° referred to the moving axes $x'o'y'$. Find the x and y components of the absolute acceleration of the particle. *Ans.* 31.52 ft. per second per second,
28.59 ft. per second per second.

14. Two motorboats having the same maximum speed of 26 ft. per second start from a pier. The first boat travels 1 mi. upstream and back at maximum speed. The second boat travels the same distance and back at right angles to the stream. If the velocity of the stream is 10 ft. per second and if each boat loses the same time in turning, show that the first boat returns to the pier $36\frac{2}{3}$ sec. later than the second boat.

15. Two concentric circular railway tracks have lengths of 30 mi. and 60 mi. A train on the inner track has a speed of 30 mi. per hour and a train on the outer track has a speed of 10 mi. per hour. Both trains move in a counterclockwise direction. Find the time in minutes that a line joining the two trains rotates in a clockwise direction.

Ans. 17.8 min.

16. Two sliders move in the same direction along parallel guides whose distance apart is s, with constant velocities v and $\frac{v}{2}$ respectively. If a rod joining the sliders makes an angle θ with the normal to the guides, and if $\theta = 0$ when $t = 0$, find the angular velocity and acceleration of the rod at any time t.

Ans. $$\frac{d\theta}{dt} = \frac{2\,sv}{4\,s^2 + v^2t^2},$$
$$\frac{d^2\theta}{dt^2} = -\frac{4\,sv^3t}{(4\,s^2 + v^2t^2)^2}.$$

140. Relative velocity (rotating axes). Let the velocity and acceleration of a particle P in plane motion be given with respect to a set of rotating rectangular axes x_roy_r whose origin and plane coincide with the origin and plane of a set of fixed rectangular axes xoy. Also let the angular velocity and angular acceleration of the rotating axes be given with respect to the fixed axes. It is required to find the velocity and acceleration of the particle P with respect to the fixed axes.

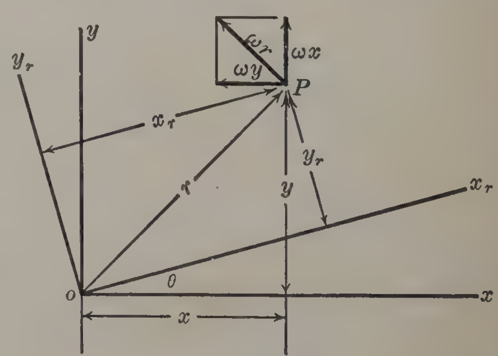

FIG. 263

Let (x_r, y_r) and (x, y) be the coördinates of the particle P referred to the rotating and fixed axes respectively, and let θ, ω, and α be the angular displacement, angular velocity, and

$x_0 = x_1 + x_0' \cos\theta$ ...

angular acceleration of the rotating axes respectively at any time t. From Fig. 263,

$$x = x_r \cos\theta - y_r \sin\theta, \tag{1}$$

$$y = x_r \sin\theta + y_r \cos\theta. \tag{2}$$

Differentiating (1) and (2) with respect to the time gives

$$\frac{dx}{dt} = \left(\frac{dx_r}{dt}\cos\theta - \frac{dy_r}{dt}\sin\theta\right) - \omega(x_r \sin\theta + y_r \cos\theta), \tag{3}$$

$$\frac{dy}{dt} = \left(\frac{dx_r}{dt}\sin\theta + \frac{dy_r}{dt}\cos\theta\right) + \omega(x_r \cos\theta - y_r \sin\theta). \tag{4}$$

These equations express the components of the absolute velocity of the particle P along the fixed axes in terms of its relative velocity, its relative position, and the angular displacement and angular velocity of the rotating axes. The following explanation of (3) and (4) shows that they may be interpreted vectorially.

The left-hand members, $\frac{dx}{dt}$ and $\frac{dy}{dt}$, are the components along the fixed axes of the *absolute-velocity vector*. The quantities $\frac{dx_r}{dt}$ and $\frac{dy_r}{dt}$ are the components along the rotating axes of the relative-velocity vector, and, by projecting them upon the fixed axes, $\left(\frac{dx_r}{dt}\cos\theta - \frac{dy_r}{dt}\sin\theta\right)$ and $\left(\frac{dx_r}{dt}\sin\theta + \frac{dy_r}{dt}\cos\theta\right)$ are the components along the fixed axes of the same *relative-velocity vector*. The terms $-\omega(x_r \sin\theta + y_r \cos\theta)$ and $\omega(x_r \cos\theta - y_r \sin\theta)$ or their equivalents, $-\omega y$ and ωx, are the components along the fixed axes of the *entrained-velocity vector* ωr.

Hence the total *absolute velocity* of the particle P is equal to the vector sum of its *relative velocity* with respect to the rotating axes and its *entrained velocity*.

This result can be inferred from the conclusions of § 136. The statement that the absolute velocity is equal to the vector sum of the relative velocity and the entrained velocity is true whether the moving axes have a motion of translation or of rotation. The statement that the absolute acceleration is equal to the vector sum of the relative acceleration and the entrained acceleration is true only when the moving axes have a motion of translation. If the moving axes rotate, the statement must be modified (§ 141).

141. Relative acceleration (rotating axes); theorem of Coriolis. Differentiating equations (3) and (4) of § 140 with respect to the time and simplifying by replacing $(x_r \cos\theta - y_r \sin\theta)$ by x and $(x_r \sin\theta + y_r \cos\theta)$ by y,

$$\frac{d^2x}{dt^2} = \left(\frac{d^2x_r}{dt^2}\cos\theta - \frac{d^2y_r}{dt^2}\sin\theta\right) - \omega^2 x - \alpha y$$
$$- 2\,\omega\left(\frac{dx_r}{dt}\sin\theta + \frac{dy_r}{dt}\cos\theta\right), \qquad (1)$$

$$\frac{d^2y}{dt^2} = \left(\frac{d^2x_r}{dt^2}\sin\theta + \frac{d^2y_r}{dt^2}\cos\theta\right) - \omega^2 y + \alpha x$$
$$+ 2\,\omega\left(\frac{dx_r}{dt}\cos\theta - \frac{dy_r}{dt}\sin\theta\right). \qquad (2)$$

These equations express the components of the absolute acceleration of the particle P along the fixed axes in terms of the relative-acceleration and relative-velocity components along the rotating axes, the angular displacement, angular velocity, and angular acceleration of the rotating axes, and the coördinates of the particle referred to the fixed axes. Equations (1) and (2) may be brought into the form of a convenient vector equation as follows:

The left-hand members, $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$, are the components along the fixed axes of the absolute-acceleration vector. The quantities $\frac{d^2x_r}{dt^2}$ and $\frac{d^2y_r}{dt^2}$ are the components along the rotating axes of the relative-acceleration vector, and by projection the quantities $\left(\frac{d^2x_r}{dt^2}\cos\theta - \frac{d^2y_r}{dt^2}\sin\theta\right)$ and $\left(\frac{d^2x_r}{dt^2}\sin\theta + \frac{d^2y_r}{dt^2}\cos\theta\right)$ are the components along the fixed axes of the same relative-acceleration vector. The terms $-\omega^2 x$ and $-\omega^2 y$ are the components along the fixed axes of the radial component of the entrained-acceleration vector $-\omega^2 r$. The terms $-\alpha y$ and αx are the components along the fixed axes of the tangential component of the entrained-acceleration vector αr. The terms $\frac{dx_r}{dt}$ and $\frac{dy_r}{dt}$ are the components along the rotating axes of the relative-velocity vector of the particle P. For convenience the relative-velocity vector is designated by u. The expressions $\left(\frac{dx_r}{dt}\sin\theta + \frac{dy_r}{dt}\cos\theta\right)$ and $\left(\frac{dx_r}{dt}\cos\theta - \frac{dy_r}{dt}\sin\theta\right)$ are by pro-

jection the components along the fixed axes of the relative-velocity vector u.

If the vectors themselves are considered rather than their components, equations (1) and (2) may be summarized as follows:

The *absolute acceleration* of a particle P is the vector sum of

1. The *relative acceleration* of the particle P with respect to the rotating axes.

2. The *entrained acceleration*, that is, the acceleration of a point fixed to the rotating axes which instantaneously coincides with the particle P.

3. The *complementary acceleration*, that is, twice the product of the angular velocity of the rotating axes and the relative velocity u of the particle P.

This theorem is known as the *theorem of Coriolis*.

The angle which the complementary acceleration makes with the fixed x axis is 90° greater in the positive trigonometric sense than the angle which the relative-velocity vector makes with the same axis. This is evident from the slopes of the two vectors, thus,

$$\frac{2\,\omega\left(\dfrac{dx_r}{dt}\cos\theta - \dfrac{dy_r}{dt}\sin\theta\right)}{-2\,\omega\left(\dfrac{dx_r}{dt}\sin\theta + \dfrac{dy_r}{dt}\cos\theta\right)} \quad \text{and} \quad \frac{\left(\dfrac{dx_r}{dt}\sin\theta + \dfrac{dy_r}{dt}\cos\theta\right)}{\left(\dfrac{dx_r}{dt}\cos\theta - \dfrac{dy_r}{dt}\sin\theta\right)}.$$

It follows that the sense of the complementary-acceleration vector is such that if it were considered as a force applied at the forward end of the relative-velocity vector it would tend to rotate the angular-velocity vector in the same sense as the axes rotate.

EXAMPLES

1. The motion of a slender tube rotating about one end in a horizontal plane is referred to a set of axes fixed in the plane. The axis of rotation passes through the origin perpendicular to the plane, and the tube makes an angle of 30° with the x axis. The angular velocity of the tube is 2 radians per second and its angular acceleration is $\frac{1}{4}$ radian per second per second. A particle within the tube at a distance of 3 ft. from the axis of rotation moves away from the axis with a velocity of $\frac{1}{3}$ ft. per second and an acceleration of $\frac{1}{10}$ ft. per second per second. Find the x and y components of the absolute acceleration of the particle.

Solution. A set of rotating axes $x_r o y_r$ is selected so that the rotating axis ox_r coincides with the axis of the tube. Then, according to the theorem of Coriolis, the absolute acceleration of the particle is the vector sum of

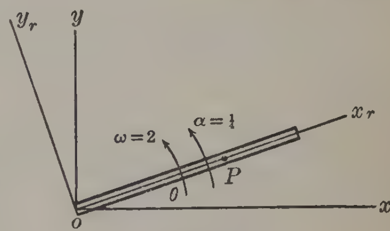

FIG. 264

a. The *relative acceleration,* that is, the acceleration of the particle referred to the tube. The components of the relative acceleration are, from the problem, $\frac{1}{10}$ ft. per second per second along the tube and zero along the y_r axis. The projections of these component accelerations on the fixed axes are

$$\tfrac{1}{10} \cos 30° \text{ along the } x \text{ axis and } \tfrac{1}{10} \sin 30° \text{ along the } y \text{ axis.}$$

b. The *entrained acceleration,* that is, the acceleration which the particle would have if it were at rest in the tube and the tube continued its rotation. In this case the entrained acceleration is simply the acceleration of a point moving in a circle. The radial component is $2^2(3)$ toward the origin, and the tangential component is $\frac{1}{4}(3)$. The projections of these accelerations on the fixed axes are

$$-12 \cos 30° - \tfrac{3}{4} \sin 30° \text{ along the } x \text{ axis}$$

and

$$-12 \sin 30° + \tfrac{3}{4} \cos 30° \text{ along the } y \text{ axis.}$$

c. The *complementary acceleration,* that is, twice the product of the angular velocity of the tube and the velocity of the particle with respect to the tube. The direction of the complementary acceleration is the direction which the relative-velocity vector would have if it were rotated through an angle of 90° in the positive trigonometric sense, that is, at right angles to the tube. The complementary acceleration is $2(2)(\frac{1}{3})$ and its projections on the fixed axes are

$$-\tfrac{4}{3} \sin 30° \text{ along the } x \text{ axis and } \tfrac{4}{3} \cos 30° \text{ along the } y \text{ axis.}$$

The x and y components of the absolute acceleration of the particle are the sums of the components obtained under (a), (b), and (c). Hence

$$\frac{d^2x}{dt^2} = \tfrac{1}{10} \cos 30° - 12 \cos 30° - \tfrac{3}{4} \sin 30° - \tfrac{4}{3} \sin 30° = -11.35 \text{ ft. per second per second,}$$

and

$$\frac{d^2y}{dt^2} = \tfrac{1}{10} \sin 30° - 12 \sin 30° + \tfrac{3}{4} \cos 30° + \tfrac{4}{3} \cos 30° = -4.15 \text{ ft. per second per second.}$$

2. The bar AB rotates in a plane about the fixed point o, with an angular velocity ω' and an angular acceleration α'. The bar BC rotates in the same plane about the end B of the first rod with an absolute angular velocity ω and absolute angular acceleration α. Find the x and y components of the absolute acceleration of the point C in the position of the bars shown in Fig. 265.

Solution. The absolute acceleration of the point C with respect to the fixed axes xoy is the vector sum of

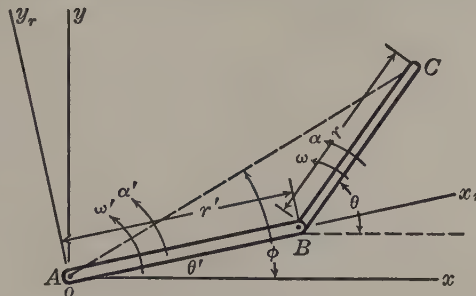

FIG. 265

a. The *relative acceleration* of the point C with respect to axes x_roy_r rotating with the bar AB. The angular velocity and angular acceleration of the bar BC with respect to the rotating axes are $(\omega - \omega')$ and $(\alpha - \alpha')$ respectively. The relative acceleration of the point C is $(\omega - \omega')^2 r$ in the direction CB and $(\alpha - \alpha')r$ perpendicular to CB. The projections of these component accelerations on the fixed axes are

$$-(\omega - \omega')^2 r \cos\theta - (\alpha - \alpha')r \sin\theta \text{ along the } x \text{ axis}$$

and $$-(\omega - \omega')^2 r \sin\theta + (\alpha - \alpha')r \cos\theta \text{ along the } y \text{ axis.}$$

b. The *entrained acceleration*, that is, the acceleration that the point C would have with respect to the fixed axes if the two bars were welded together at B. The components of the entrained acceleration are $(\omega')^2(CA)$ along CA and $\alpha'(CA)$ perpendicular to CA. Their projections on the fixed axes are

$$-(\omega')^2(CA)\cos\phi - \alpha'(CA)\sin\phi \text{ along the } x \text{ axis}$$

and $$-(\omega')^2(CA)\sin\phi + \alpha'(CA)\cos\phi \text{ along the } y \text{ axis.}$$

Since $$\sin\phi = \frac{r'\sin\theta' + r\sin\theta}{CA}$$

and $$\cos\phi = \frac{r'\cos\theta' + r\cos\theta}{CA},$$

the components along the fixed axes become

$$-(\omega')^2(r'\cos\theta' + r\cos\theta) - \alpha'(r'\sin\theta' + r\sin\theta) \text{ along the } x \text{ axis}$$

and $$-(\omega')^2(r'\sin\theta' + r\sin\theta) + \alpha'(r'\cos\theta' + r\cos\theta) \text{ along the } y \text{ axis.}$$

c. The *complementary acceleration*, that is, twice the product of the angular velocity of the bar AB and the velocity of the point C with respect to the axes x_roy_r. The relative velocity of the point C is $(\omega - \omega')r$ perpendicular to the bar BC. Hence the complementary acceleration is $2\,\omega'(\omega - \omega')r$ in the direction CB. Its components along the fixed axes are

$$-2\,\omega'(\omega - \omega')r\cos\theta \text{ along the } x \text{ axis}$$

and $$-2\,\omega'(\omega - \omega')r\sin\theta \text{ along the } y \text{ axis.}$$

The x and y components of the absolute acceleration of the point C are the sums of the components obtained under (a), (b), and (c). Hence, adding and simplifying,

$$\frac{d^2x}{dt^2} = -\omega'^2 r'\cos\theta' - \omega^2 r\cos\theta - \alpha' r'\sin\theta' - \alpha r\sin\theta,$$

and $$\frac{d^2y}{dt^2} = -\omega'^2 r'\sin\theta' - \omega^2 r\sin\theta + \alpha' r'\cos\theta' + \alpha r\cos\theta.$$

3. A wheel of radius r rolls on the outside of a fixed wheel of radius R with an absolute angular velocity ω and an absolute angular acceleration α. Find the absolute acceleration of any point on the rim of the rolling wheel.

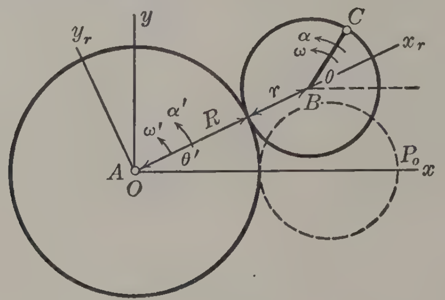

FIG. 266

Solution. From Example 2, p. 207, the components of the absolute acceleration of the point C are

$$\frac{d^2x}{dt^2} = -\omega'^2 r' \cos\theta' - \omega^2 r \cos\theta - \alpha' r' \sin\theta' - \alpha r \sin\theta,$$

$$\frac{d^2y}{dt^2} = -\omega'^2 r' \sin\theta' - \omega^2 r \sin\theta + \alpha' r' \cos\theta' + \alpha r \cos\theta.$$

If the wheel of radius r rolls without slipping on the wheel of radius R, the angle θ described by the radius BC and the angle θ' described by the radius R are connected by the relation

$$R\theta' = r(\theta - \theta'), \quad \text{or} \quad \theta' = \frac{r\theta}{R+r}.$$

Differentiating with respect to the time gives

$$\omega' = \frac{r\omega}{R+r} \quad \text{and} \quad \alpha' = \frac{r\alpha}{R+r}.$$

Also $R + r$ is equal to the r' of Example 2.

Making use of these relations, the first two equations become

$$\frac{d^2x}{dt^2} = -\omega^2 r\left(\frac{r}{R+r}\cos\frac{r\theta}{R+r} + \cos\theta\right) - \alpha r\left(\sin\frac{r\theta}{R+r} + \sin\theta\right),$$

$$\frac{d^2y}{dt^2} = -\omega^2 r\left(\frac{r}{R+r}\sin\frac{r\theta}{R+r} + \sin\theta\right) + \alpha r\left(\cos\frac{r\theta}{R+r} + \cos\theta\right).$$

PROBLEMS

1. Use the theorem of Coriolis to derive the radial and transverse components of the acceleration of a point describing a plane curve.

2. A straight tube AB rotates in a plane about the end A with an angular velocity of 3 radians per second and an angular acceleration of 4 radians per second per second. A particle within the tube at a distance of 2 ft. from A moves toward B with a velocity relative to

the tube of 6 ft. per second and an acceleration toward A of 3 ft. per second per second. Find the absolute acceleration of the particle.

Ans. 48.76 ft. per second per second, 115° 31′.

3. A wheel of radius r rolls on the outside of a fixed wheel of radius R with an absolute angular velocity ω and an absolute angular acceleration α. Find the x and y components of the absolute acceleration of the points A, B, C, and D.

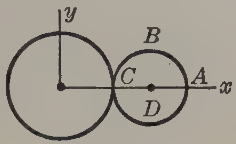

FIG. 267

Ans.

	A	B	C	D
$\dfrac{d^2x}{dt^2}=$	$-\dfrac{r(R+2r)\omega^2}{R+r}$	$-\dfrac{r^2\omega^2}{R+r}-r\alpha$	$\dfrac{rR\omega^2}{R+r}$	$-\dfrac{r^2\omega^2}{R+r}+r\alpha$
$\dfrac{d^2y}{dt^2}=$	$2r\alpha$	$-r\omega^2+r\alpha$	0	$r\omega^2+r\alpha$

4. A wheel of radius r rolls on the inside of a fixed wheel of radius R with an absolute angular velocity ω and an absolute angular acceleration α. Find the x and y components of the absolute acceleration of the points A, B, C, and D.

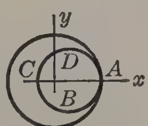

FIG. 268

Ans.

	A	B	C	D
$\dfrac{d^2x}{dt^2}=$	$-\dfrac{Rr\omega^2}{R-r}$	$-\dfrac{r^2\omega^2}{R-r}+r\alpha$	$\dfrac{r(R-2r)}{R-r}\omega^2$	$-\dfrac{r^2\omega^2}{R-r}-r\alpha$
$\dfrac{d^2y}{dt^2}=$	0	$r\omega^2-r\alpha$	$-2r\alpha$	$-r\omega^2-r\alpha$

5. Determine expressions for the components of the absolute velocity and absolute acceleration of the particle P of §§ 140 and 141 at the instant when the rotating x axis is passing through the fixed x axis.

$$\textit{Ans.}\ \frac{dx}{dt}=\frac{dx_r}{dt}-\omega y_r,\quad \frac{d^2x}{dt^2}=\frac{d^2x_r}{dt^2}-\omega^2x_r-\alpha y_r-2\,\omega\frac{dy_r}{dt},$$
$$\frac{dy}{dt}=\frac{dy_r}{dt}+\omega x_r,\quad \frac{d^2y}{dt^2}=\frac{d^2y_r}{dt^2}-\omega^2y_r+\alpha x_r+2\,\omega\frac{dx_r}{dt}.$$

6. Solve Examples 1, 2, and 3 on pages 205–208 by means of the simplified equations of Problem 5, above.

7. What are the necessary conditions that the complementary acceleration of Coriolis may vanish?

8. The coördinates, velocity, and acceleration $\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)$ of a point referred to fixed axes, and the angle, angular velocity, and angular acceleration (θ, ω, α) of a set of rotating axes whose origin coincides with the origin of the fixed axes are given. Show that the velocity and acceleration of the point referred to the rotating axes are

$$\frac{dx_r}{dt} = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta - \omega x\sin\theta + \omega y\cos\theta,$$

$$\frac{dy_r}{dt} = \frac{dy}{dt}\cos\theta - \frac{dx}{dt}\sin\theta - \omega y\sin\theta - \omega x\cos\theta,$$

$$\frac{d^2x_r}{dt^2} = \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta - 2\,\omega\frac{dx}{dt}\sin\theta + 2\,\omega\frac{dy}{dt}\cos\theta$$
$$- \omega^2 x\cos\theta - \omega^2 y\sin\theta - \alpha x\sin\theta + \alpha y\cos\theta,$$

$$\frac{d^2y_r}{dt^2} = \frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta - 2\,\omega\frac{dy}{dt}\sin\theta - 2\,\omega\frac{dx}{dt}\cos\theta$$
$$+ \omega^2 x\sin\theta - \omega^2 y\cos\theta - \alpha x\cos\theta - \alpha y\sin\theta.$$

DYNAMICS

CHAPTER XIV

RECTILINEAR MOTION

142. Equation of motion. The science of dynamics has for its object the investigation of the motion of bodies under the action of forces. If the motion of a particle and the force influencing its motion are in the same straight line, the relation between the force F, the mass m, and the acceleration a of the particle is expressed by the equation (§ 10)

$$F = ma. \tag{1}$$

In a later chapter it is shown that this equation is also valid for a body.

If the motion is along the x axis and if the positive sense of the force and the positive sense of the acceleration are chosen in the positive direction of x, (1) may be written

$$F = m\frac{d^2x}{dt^2}, \tag{2}$$

or

$$F = m\frac{dv}{dt}, \tag{3}$$

or, since

$$\frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx},$$

$$F = mv\frac{dv}{dx}. \tag{4}$$

EXAMPLES

1. A body weighing 20 lb. rests upon a rough horizontal plane ($\mu = 0.2$). Find the horizontal force necessary to overcome the friction and to give the body an acceleration of 3.22 ft. per second per second.

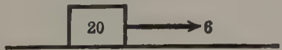

FIG. 269

Solution. The force of friction is $0.2 \times 20 = 4$ lb. If F is the required force, then $F - 4$ is the force necessary for acceleration alone. Hence

$$F - 4 = \frac{20}{32.2}(3.22), \quad \text{from which} \quad F = 6 \text{ lb.}$$

2. A body weighing 20 lb. is suspended by a cord. If the tension in the cord is 30 lb., find the acceleration of the body.

Solution. Since gravity exerts a downward force of 20 lb. on the body, the resultant force causing acceleration is upward and equal to 10 lb. The acceleration is therefore upward and it is found from the equation

$$(30 - 20) = 10 = \frac{20}{32.2}\,a.$$

Hence $a = 16.1$ ft. per second per second.

3. A body A resting on a rough inclined plane $(\mu = \frac{3}{10})$ is connected to a body B resting on a rough horizontal plane by a light cord which passes over a peg C as shown in Fig. 270. Find the acceleration of the bodies and the tensions in the cord if the friction of the peg on the cord is 3 lb.

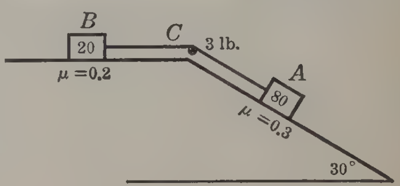

FIG. 270

Solution. If T is the tension in the lower segment of the cord, $T - 3$ is the tension in the upper segment. The forces which act on the 80-pound body parallel to the inclined plane are $80 \sin 30°$, $-0.3(80 \cos 30°)$, and $-T$. Hence its equation of motion is

$$80 \sin 30° - 0.3(80 \cos 30°) - T = \frac{80}{32.2}\,a.$$

Similarly the equation of motion of the 20-pound body is

$$T - 3 - 0.2(20) = \frac{20}{32.2}\,a.$$

Solving these equations,

$$T = 9.44 \text{ lb.} \quad \text{and} \quad a = 3.93 \text{ ft. per second per second.}$$

In this problem, if there is any motion, its direction is evident. If a third body D were attached to the body B by a cord running over a peg at the left of B, the direction of motion of the system might not be evident. An incorrect assumption of the direction of motion when energy is lost through friction leads to incorrect results. The direction or absence of motion should be first obtained by the methods of statics.

PROBLEMS

1. An elevator cage with its load weighs 1 T. The force of friction of the cage upon the guides is constant and equal to 200 lb. Find the force exerted by the cable (*a*) when the acceleration of the cage is 4 ft. per second per second upward, (*b*) when the acceleration is 4 ft. per second per second downward. *Ans.* 2448 lb., 1552 lb.

2. A spring scale suspended from the ceiling of an elevator cage carries a weight of 40 lb. What will the scale read (*a*) when the cage is accelerated upward at the rate of 5 ft. per second per second?

(*b*) when the cage has a downward acceleration of 8 ft. per second per second? Neglect the inertia of the scale spring and hook.

Ans. 46.28 lb., 30.05 lb.

3. A body A weighing 8 lb., which rests on a smooth horizontal plane, is attached to a second body B, weighing 16 lb., by a string passing over a smooth peg C. Find the acceleration of the body B and the tension in the string.

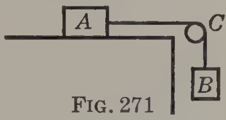

FIG. 271

Ans. $a = 21.5$ ft. per second per second, $T = 5.32$ lb.

4. Find the tension in the cord and the acceleration of the weights for the system shown in Fig. 272. Neglect the weights of the pulleys.

HINT. The tension in the cord is the same in all its parts.

Ans. $a = 21.47$ ft. per second per second,
$a = 10.73$ ft. per second per second,
$T = 5.33$ lb.

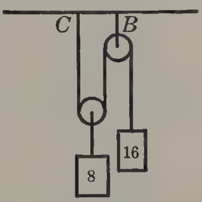

FIG. 272

5. A string carrying weights of 8 lb. and 16 lb. at its ends passes over two smooth fixed pegs, A and B, and through a smooth movable ring C weighing 6 lb. Find the acceleration of the weights C, D, and E and the tension in all parts of the cord.

Ans. 18.06 ft. per second per second,
13.35 ft. per second per second,
22.78 ft. per second per second,
$T = 4.68$ lb.

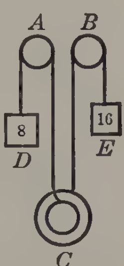

FIG. 273

6. A weight A of 20 lb. resting on a rough horizontal plane ($\mu = 0.2$) is attached to one end of a string which passes over a smooth peg D and through a ring C weighing 40 lb. The other end of the string is attached to the ceiling at B as shown in Fig. 274. Find the acceleration of the weight A and the tension in the string.

Ans. 17.16 ft. per second per second, 14.67 lb.

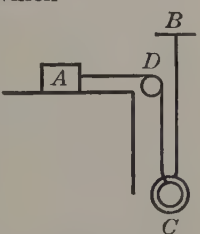

FIG. 274

7. The drawbar pull of a locomotive is 40,000 lb. The weight of the train, excluding the locomotive, is 2000 T., and the train resistance is 15 lb. per ton. Find the acceleration imparted to the train when running on a level track. *Ans.* 0.0805 ft. per second per second.

8. A balloon of weight W falls with an acceleration f. How much ballast must be thrown out in order that it may have an upward acceleration f, air resistance being neglected? *Ans.* $\dfrac{2Wf}{f+g}$.

9. Two weights, A and B, attached together by a cord as shown in Fig. 275, are acted upon by the force of gravity and a vertical tension $T_1 = 10$ lb. Find the acceleration of the weights and the tension T_2.

Ans. $a = 23.26$ ft. per second per second, $T_2 = 4.44$ lb.

10. A spring scale weighing 2 lb. carries a weight of 8 lb. The scale is suspended in an elevator cage weighing 1000 lb. If the scale reads 12 lb. when the elevator is in motion, determine the pull of the lifting cable on the cage.

Ans. 1515 lb.

11. A body weighing 36 lb. is broken into two parts which are attached to the ends of a cord running over a smooth fixed horizontal cylinder. If the acceleration of each part is $0.05\,g$, find the weight of each part. *Ans.* 17.1 lb., 18.9 lb.

FIG. 275

143. Motion in a straight line under constant force. The equation of motion is

$$F = ma \tag{1}$$

and, since F and m are assumed constant, the acceleration is constant. Hence the equation of motion becomes

$$\frac{d^2x}{dt^2} = a = \frac{F}{m}. \tag{2}$$

Integrating with respect to time,

$$v = \frac{dx}{dt} = at + c_1. \tag{3}$$

Integrating again,
$$x = \frac{at^2}{2} + c_1t + c_2. \tag{4}$$

Let the initial velocity be v_0, so that $v = v_0$ when $x = 0$ and $t = 0$. The constants c_1 and c_2 are obtained by substituting the initial conditions in (3) and (4). Hence

$$v = v_0 + at \tag{5}$$

and
$$x = v_0t + \frac{at^2}{2}. \tag{6}$$

Eliminating t between (5) and (6),

$$v^2 = v_0^2 + 2\,ax. \tag{7}$$

Equations (5), (6), and (7) are valid for the motion of a falling body near the surface of the earth if the acceleration of gravity g is substituted for a, and if the positive direction of x is taken downward.

PROBLEMS

1. A body is projected upward with a speed of 100 ft. per second. Find the time during which the body rises and the distance traversed during that time, air resistance being neglected.

Ans. 3.11 sec., 155.3 ft.

2. A body is projected vertically downward with a speed of 100 ft. per second from a height of 60 ft. Find the speed at the end of the fall and the time of descent. *Ans.* 117.7 ft. per second, 0.55 sec.

3. A particle which is projected vertically upward is observed to rise 100 ft. Find the initial velocity and the total time of flight.

Ans. 80.25 ft. per second, 4.98 sec.

4. Show that equations (5), (6), and (7) on page 214 are valid for a particle sliding down a smooth plane inclined at an angle θ to the horizontal, provided $g \sin \theta$ is substituted for a.

5. A stone is dropped into a well and the sound of the splash is heard after 3 sec. Find the depth of the well, the velocity of sound being 1100 ft. per second. Interpret the second root of the quadratic equation. *Ans.* 133 ft.

6. A body is projected vertically downward with an initial speed of 30 ft. per second from the top of a tower 100 ft. high. At the same instant a second body is projected vertically upward from the foot of the tower with a speed of 100 ft. per second. How far are the bodies from the top of the tower when they meet, and what time is required? *Ans.* 32.6 ft., $t = \frac{10}{13}$ sec.

7. A boy throws a ball vertically upward with a velocity of 40 ft. per second, and after one second he throws a second ball upward with a velocity of 30 ft. per second. When and where do the balls pass each other? *Ans.* 2.075 sec., 13.57 ft.

8. A body is projected up a plane inclined at 45° to the horizontal with a speed of 40 ft. per second. If the coefficient of friction between the body and the plane is 0.2, find the distance the body will travel up the plane. Find the time of going up and returning to the starting point. Find the speed with which the body returns to the starting point. *Ans.* 29.3 ft., 3.25 sec., 32.7 ft. per second.

9. Two bodies, A and B, weighing 10 lb. and 30 lb. respectively, rest upon rough inclined planes, Fig. 276. The bodies are connected by a string passing over a smooth peg C. Find the acceleration of either body if the coefficient of friction is 0.3. Also find the distance the body B moves from rest in 4 sec.

Ans. 11.2 ft. per second per second, 89.6 ft.

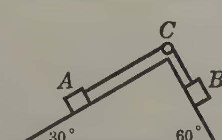

FIG. 276

10. A body falling freely from rest enters a room 12 ft. high through a hole in the ceiling and passes out through a hole in the floor $\frac{1}{5}$ sec. later. How high is the starting point above the floor?
Ans. 62 ft.

11. A train 1000 ft. long moves with constant acceleration. The pilot passes a certain milepost at a speed of 30 mi. per hour, and the rear end of the train passes the same milepost with a speed of 60 mi. per hour. Find the point in the train which passes the milepost at 45 mi. per hour. *Ans.* A point 415 ft. from the pilot.

12. Two particles starting from rest move along a straight line in the same direction. The first particle has an acceleration of 4 ft. per second per second and it begins its motion 8 sec. before the second particle, which has an acceleration of 6 ft. per second per second. How long after the first particle starts will the second particle overtake it, and what is then their distance from the starting point? Explain the meaning of the second root of the quadratic equation. *Ans.* 43.59 sec., 3800 ft.

13. A train having a constant acceleration of 2 ft. per second per second attains a speed of 60 mi. per hour from rest. The train then runs at uniform speed until nearing its destination, when an application of the brakes produces a constant retardation of 4 ft. per second per second and brings it to rest. The total distance being 2 mi., find the time of the journey. *Ans.* 153 sec.

14. The breaking load of a cable supporting an elevator cage is 8000 lb. The factor of safety is 4. The cage with its load weighs 1500 lb. Find the minimum time required to lift the loaded cage a distance of 16 ft. from rest if the cable is not stressed beyond its proper working strength. *Ans.* 1.73 sec.

15. A body weighing 40 lb. is at rest on a rough horizontal plane. ($\mu = 0.3$.) Find the constant horizontal force necessary to give it a velocity of 30 ft. per second at the end of 3 sec. *Ans.* 24.42 lb.

16. A body weighing 20 lb. acquires a velocity of 60 ft. per second from rest in 3 sec. under the action of a constant force. Determine the force. *Ans.* 12.42 lb.

17. A train of 10 coaches weighing 80 T. each is uniformly accelerated by a locomotive so that a speed of 40 mi. per hour is attained in 6 min. from rest. The train resistance, excluding air resistance, is 6 lb. per ton. The air resistance, which varies as the square of the speed, is 360 lb. at 40 mi. per hour. Find the drawbar pull of the locomotive when the speed is 20 mi. per hour.
Ans. 5700 lb.

18. A weight of 10 lb. is attached to each end of a cord running over a smooth fixed horizontal cylinder. Find the weight which must be added to one of the weights to cause it to fall a distance of 10 ft. from rest in 5 sec. *Ans.* $\frac{80}{157}$ lb.

19. A particle falls one fifth of the total distance fallen during the last second of its fall. Find the total time of the fall.

Ans. $(5 + 2\sqrt{5})$ sec.

144. The energy equation; the momentum equation. The equation

$$F = m\frac{d^2x}{dt^2} \tag{1}$$

admits of two first integrals which are of fundamental importance.

The *energy equation* is obtained by multiplying both members of (1) by dx and integrating with respect to x. Thus

$$\int_{x_0}^{x} F\,dx = \int_{x_0}^{x} m\frac{d^2x}{dt^2}\,dx = \int_{v_0}^{v} mv\,dv$$

from equation (4), § 142.

Hence
$$\int_{x_0}^{x} F\,dx = \frac{mv^2}{2} - \frac{mv_0^2}{2}, \tag{2}$$

where v_0 and x_0 are the values of v and x, respectively, when $t = t_0$.

The *momentum equation* is obtained by multiplying both members of (1) by dt and integrating with respect to t. Thus

$$\int_{t_0}^{t} F\,dt = \int_{t_0}^{t} m\frac{d^2x}{dt^2}\,dt = \int_{v_0}^{v} m\,dv$$

from equation (3), § 142.

Hence
$$\int_{t_0}^{t} F\,dt = mv - mv_0. \tag{3}$$

The first member of (2) is the work done by the force F in moving from x_0 to x (§ 92). If F is constant the integral becomes $F(x - x_0)$. The expression $\frac{mv^2}{2}$ is called the *kinetic energy* of the mass m moving with the velocity v. Equation (2) states that the change in kinetic energy of any mass or body moving in a straight line is equal to the work done by the external force acting upon the body while its velocity is changed from v_0 to v.

The first member of (3) is called the *impulse* of the force F acting through the time interval $t - t_0$. The quantity mv has

been defined as the momentum of the mass m moving with velocity v (§ 10). Equation (3) states that the change in momentum of any mass or body moving in a straight line is equal to the impulse of the external force acting upon the body while its velocity is changed from v_0 to v or during the time interval $t - t_0$.

EXAMPLES

1. A body weighing 10 lb. moves along a rough horizontal plane ($\mu = 0.2$) under the action of a horizontal force of 4 lb. If the initial velocity of the body is 12 ft. per second, find the velocity after the body has moved 60 ft.

Solution. The friction is $(0.2)(10) = 2$ lb. The force available for producing kinetic energy is the total force less the force expended in friction, or $(4 - 2)$ lb. $= 2$ lb. The work done by the 2-pound force is $(2)(60) = 120$ ft.-lb. The initial kinetic energy is

$$\frac{m}{2} v_0^2 = \frac{10}{64.4}(144).$$

Therefore
$$120 = \frac{10}{64.4} v^2 - \frac{10}{64.4}(144),$$

from which
$$v = 30.28 \text{ ft. per second.}$$

2. A body weighing 8 lb. moves along a horizontal straight line with a velocity of 20 ft. per second. Find the kinetic energy of the body and the vertical distance the kinetic energy would raise the body against the force of gravity if it were all available for that purpose.

Solution. The kinetic energy of the body is

$$\frac{1}{2} mv^2 = \frac{1}{2}\left(\frac{8}{32.2}\right)(20)^2 = 49.7 \text{ ft.-lb.}$$

The vertical distance raised is therefore

$$\frac{49.7}{8} = 6.21 \text{ ft.}$$

3. A boy exerts a constant force of 6 lb. on a ball weighing 8 oz. for $\frac{1}{5}$ sec. Find the velocity of the ball.

Solution. The impulse of the force acting on the ball is

$$Ft = (6)(\tfrac{1}{5}) = \tfrac{6}{5} \text{ lb.-sec.}$$

The initial momentum of the ball is $mv_0 = 0$. The final momentum is

$$mv = \frac{8}{(16)32.2} v.$$

Equating the impulse to the change in momentum,

$$\frac{6}{5} = \frac{8}{(16)(32.2)} v,$$

or
$$v = 77.3 \text{ ft. per second.}$$

4. A locomotive exerting a constant drawbar pull changes the speed of a train weighing 200 T. from 30 mi. per hour to 60 mi. per hour in 3 min. on a level track. The train resistance is a constant force of 10 lb. per ton. Find the drawbar pull of the locomotive.

Solution. The total train resistance is $(200)(10) = 2000$ lb. The impulse of the force F which changes the momentum of the train is $F(3 \times 60)$ lb.-sec. The change of momentum of the train is

$$\frac{(200)(2000)}{32.2}(88 - 44) \text{ lb.-sec.}$$

Equating the impulse and change of momentum,

$$180\,F = \frac{400{,}000(44)}{32.2},$$

from which $F = 3037$ lb. Hence the drawbar pull is $3037 + 2000 = 5037$ lb.

PROBLEMS

1. A body weighing 10 lb. is moving along a horizontal plane with a velocity of 40 ft. per second. Find the distance the body will move before coming to rest. ($\mu = 0.2$.) *Ans.* 124 ft.

2. The hammer of a pile-driver weighing $\frac{1}{2}$ T. falls 16 ft. against a constant frictional resistance of 200 lb. Find its striking velocity. *Ans.* 14.4 ft. per second.

3. A body weighing 20 lb. is suspended by a spring whose modulus is 5 lb. per inch. Find the total work stored in the spring when the weight is drawn down a distance of 6 in. below its equilibrium position. The force being removed, find the velocity with which the body passes a point 2 in. above the lowest position. *Ans.* 250 in.-lb., 3.66 ft. per second.

4. A body weighing 40 lb. is suspended by a spring whose modulus is 10 lb. per inch. If the body is attached to the unstretched spring and allowed to drop, find its maximum velocity. *Ans.* $\sqrt{\frac{g}{3}}$.

5. A weight of 32.2 lb. rests upon a horizontal plane. ($\mu = 0.3$.) Find the velocity it acquires after moving from rest a distance of 10 ft. under the action of a horizontal force which is expressed by $(3\,x + 10)$ lb., assuming that $x = 0$ at the beginning of the motion. *Ans.* 17.5 ft. per second.

6. Find the work performed by a locomotive which changes the speed of a train weighing 200 T. from 30 mi. per hour to 60 mi. per hour in a distance of 5000 ft. on a level track, if train resistance is considered constant and equal to 15 lb. per ton. *Ans.* 51,070,000 ft.-lb.

7. A freight car starts from rest and descends a 2-per-cent grade a distance of 1000 ft. Find the speed of the freight car at the bottom of the grade if the train resistance is constant and equal to 15 lb. per ton. *Ans.* 28.4 ft. per second.

8. A train weighing 322 T. runs at a speed of 60 mi. per hour. The brakes are applied and the train comes to rest in a distance of 1000 ft. If train resistance is constant and equal to 20 lb. per ton, find the total retarding force of the brakes. *Ans.* 71,000 lb.

9. A locomotive which is assumed to be capable of exerting a constant drawbar pull of 30,000 lb. is pulling a train of 10 cars weighing 70 T. each. The train has a speed of 60 mi. per hour at the foot of a 2-per-cent grade 1 mi. long. Find the speed of the train as it passes the top of the hill. Find the minimum speed at the bottom of the hill in order that the train may just reach the summit. Train resistance is assumed to be 15 lb. per ton.

Ans. 75.4 ft. per second, 45.4 ft. per second.

145. Variable force or acceleration. In the general case force, mass, or acceleration, or all of them, may be variable. They may also be functions of each other or of the velocity, time, or distance. The differential equation in some cases may easily be solved. The following cases illustrate the usual procedure:

CASE I. *The force a function of the time only.* The equation of motion becomes

$$m\frac{d^2x}{dt^2} = f(t).$$

Integrating with respect to t,

$$v = \frac{dx}{dt} = \frac{1}{m}\int f(t)dt + c_1.$$

Integrating again with respect to t,

$$x = \frac{1}{m}\int\int f(t)dt\,dt + c_1t + c_2.$$

Thus the velocity and position of the body are determined in terms of the time when sufficient data concerning the initial conditions are available to determine the constants c_1 and c_2.

CASE II. *The force a function of the distance only.* The equation of motion is

$$m\frac{d^2x}{dt^2} = f(x).$$

Multiplying both members by $2\frac{dx}{dt}$,

$$2\,m\frac{d^2x}{dt^2}\cdot\frac{dx}{dt} = 2\,f(x)\frac{dx}{dt}.$$

Integrating with respect to t,

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{m}\int f(x)dx + c_1.$$

Hence $$v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{\int f(x)dx + c_1}.$$

Solving for dt, $$dt = \pm\sqrt{\frac{m}{2}}\frac{dx}{\sqrt{\int f(x)dx + c_1}}.$$

Integrating each member,

$$t = \pm\sqrt{\frac{m}{2}}\int\frac{dx}{\sqrt{\int f(x)dx + c_1}} + c_2.$$

As before, sufficient initial conditions must be given to determine the constants and algebraic sign.

CASE III. *The force a function of the velocity only.* The equation of motion is

$$m\frac{d^2x}{dt^2} = f(v) \quad \text{or} \quad m\frac{dv}{dt} = f(v). \tag{1}$$

Solving for dt, $$dt = m\frac{dv}{f(v)}. \tag{2}$$

Integrating each member,

$$t = m\int\frac{dv}{f(v)} + c_1. \tag{3}$$

Calling the second member $\phi(v)$, the equation becomes

$$t = \phi(v). \tag{4}$$

Solving for v, $$v = \phi^{-1}(t) \quad \text{or} \quad \frac{dx}{dt} = \phi^{-1}(t), \tag{5}$$

whence, by integration,

$$x = \int\phi^{-1}(t)dt + c_2. \tag{6}$$

In case the equation $t = \phi(v)$ cannot be solved for v, the elimination of dt between (2) and the equation $v = \dfrac{dx}{dt}$ gives

$$m\frac{v\,dv}{f(v)} = dx, \tag{7}$$

from which $$x = m\int\frac{v\,dv}{f(v)} + c_2. \tag{8}$$

EXAMPLES

1. A body weighing 50 lb. starts from rest and moves along a frictionless horizontal plane under the action of a horizontal force F which varies with the time according to the relation

$$F = 5 + t + \frac{t^2}{4}.$$

Determine the motion.

Solution. The equation of motion is

$$5 + t + \frac{t^2}{4} = \frac{50}{g}\left(\frac{d^2x}{dt^2}\right),$$

from which
$$5\,t + \frac{t^2}{2} + \frac{t^3}{12} + c_1 = \frac{50}{g}\left(\frac{dx}{dt}\right).$$

Since the velocity is zero when $t = 0$, $c_1 = 0$; and hence the velocity at any time t is given by

$$v = \frac{gt}{50}\left(5 + \frac{t}{2} + \frac{t^2}{12}\right).$$

Integrating again,
$$\frac{5\,t^2}{2} + \frac{t^3}{6} + \frac{t^4}{48} + c_2 = \frac{50}{g}\,x.$$

Also $c_2 = 0$, and the distance x traversed in time t is

$$x = \frac{gt^2}{50}\left(\frac{5}{2} + \frac{t}{6} + \frac{t^2}{48}\right).$$

2. A body of weight W falls from rest at a height above the earth's surface equal to the radius of the earth. Find the time taken to fall to the earth's surface and the velocity with which it strikes, assuming the earth to be a fixed body and that the force of attraction varies inversely as the square of the distance from the center of the earth.

Solution. Let the origin be taken at the center of the earth, and let the x axis be coincident with the line joining the weight and the center of the earth. The force F acting on the weight when it is at a distance x is

$$F = -\frac{K}{x^2}.$$

Also $F = -W$ when $x = R$, the radius of the earth, and hence $K = WR^2$.

The equation of motion is

$$\frac{W}{g}\frac{d^2x}{dt^2} = -\frac{WR^2}{x^2}. \tag{1}$$

Dividing (1) by W and multiplying by $2\,g\,\frac{dx}{dt}$ and integrating

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\,gR^2}{x} + c_1. \tag{2}$$

From the initial conditions, $\frac{dx}{dt} = 0$ when $x = 2R$, and hence $c_1 = -gR$, and (2) becomes

$$v^2 = \left(\frac{dx}{dt}\right)^2 = \frac{2gR^2}{x} - gR. \tag{3}$$

Hence
$$v = \frac{dx}{dt} = -\sqrt{gR}\sqrt{\frac{2R}{x} - 1}. \tag{4}$$

The negative sign is chosen because the velocity is directed toward the origin.

When the weight strikes the earth, $x = R$ and

$$v = -\sqrt{gR} = -\sqrt{32.2 \times 20{,}900{,}000} = 25{,}900 \text{ ft. per second.}$$

Solving (4) for dt,
$$dt = -\frac{1}{\sqrt{gR}} \frac{dx}{\sqrt{\frac{2R}{x} - 1}}.$$

Hence
$$t = -\frac{1}{\sqrt{gR}} \int_{2R}^{R} \frac{\sqrt{x}\,dx}{\sqrt{2R - x}}.$$

The integral may be simplified by making the substitution

$$x = 2R\cos^2\theta.$$

Hence
$$t = \sqrt{\frac{R}{g}} \int_0^{\frac{\pi}{4}} (1 + \cos 2\theta)\, d(2\theta),$$

or
$$t = \sqrt{\frac{R}{g}}\left(1 + \frac{\pi}{2}\right) = 33 \text{ min.}$$

3. A particle is projected with a velocity V in a medium whose resistance is proportional to the velocity. Find the time for the particle to come to rest. Also find the distance traversed.

Solution. If the origin is taken at the point of projection and the particle moves in the positive direction along the x axis, the equation of motion is

$$m\frac{d^2x}{dt^2} = -kv, \quad \text{or} \quad m\frac{dv}{dt} = -kv. \tag{1}$$

Separating the variables,

$$\frac{dv}{v} = -\frac{k}{m}\,dt,$$

and integrating,
$$\log v = c_1 - \frac{kt}{m}. \tag{2}$$

Since $v = V$ when $t = 0$, therefore $c_1 = \log V$ and (2) becomes

$$\log\frac{v}{V} = -\frac{kt}{m},$$

or
$$v = Ve^{-\frac{kt}{m}}, \tag{3}$$

from which it is evident that $t = \infty$ when $v = 0$, or the particle requires infinite time to come to rest.

From (3), $$dx = Ve^{-\frac{kt}{m}}\,dt,$$

or $$x = c_2 - \frac{Vm}{k}e^{-\frac{kt}{m}}.$$

Since $x = 0$ when $t = 0$, $c_2 = \dfrac{Vm}{k}$

and therefore $$x = \frac{Vm}{k}\left(1 - e^{-\frac{kt}{m}}\right). \tag{4}$$

The distance traveled by the particle while it is coming to rest is obtained by making $t = \infty$ in (4). Hence

$$x = \frac{mV}{k}.$$

4. An airplane or any body falls under the influence of gravity in air whose resistance is assumed proportional to the square of the velocity. Investigate the motion.

Solution. If the body is at the origin when $t = 0$ and it moves in the positive direction along the x axis, the equation of motion is

$$\frac{W}{g}\frac{dv}{dt} = W - kv^2. \tag{1}$$

The accelerating force, $W - kv^2$, is approximately W for small velocities. As the velocity increases, $W - kv^2$ decreases. The critical velocity for which $W - kv^2 = 0$ is called the *terminal velocity* and is designated by V, so that

$$V = \sqrt{\frac{W}{k}}. \tag{2}$$

It is evident that V depends on the weight and the proportionality factor. The proportionality factor depends upon the shape of the airplane or body. The velocity of the airplane or body during the fall approaches the terminal velocity.

Substituting the value of k from (2) in (1) and reducing,

$$\frac{dv}{V^2 - v^2} = \frac{g}{V^2}\,dt. \tag{3}$$

Integrating and determining the constant from the condition that $v = v_0$ when $t = 0$, gives

$$\log\left[\frac{V+v}{V-v}\cdot\frac{V-v_0}{V+v_0}\right] = \frac{2\,gt}{V}, \tag{4}$$

from which $$v = V\left[\frac{(V+v_0) - (V-v_0)e^{-\frac{2\,gt}{V}}}{(V+v_0) + (V-v_0)e^{-\frac{2\,gt}{V}}}\right]. \tag{5}$$

Equation (5) expresses the velocity after any time in terms of the initial velocity v_0, the terminal velocity V, and the time. As t increases, the second term in both numerator and denominator approaches zero and the velocity v approaches the terminal velocity V.

The relation between the distance fallen and the velocity may be obtained from (3) as follows:

From
$$\frac{dv}{V^2 - v^2} = \frac{g}{V^2}\,dt,$$

$$\frac{1}{V^2 - v^2} \cdot \frac{dv}{dt} = \frac{g}{V^2}.$$

Multiplying each member by dx,

$$\frac{a\,dx}{V^2 - v^2} = \frac{g\,dx}{V^2},$$

and since $a\,dx = v\,dv$,
$$\frac{v\,dv}{V^2 - v^2} = \frac{g\,dx}{V^2}. \tag{6}$$

Integrating and determining the constant from the condition that $x = 0$ when $v = v_0$,

$$\log \frac{V^2 - v^2}{V^2 - v_0{}^2} = -\frac{2\,gx}{V^2}. \tag{7}$$

Solving for v^2,
$$v^2 = V^2\left[1 - \left(1 - \frac{v_0{}^2}{V^2}\right)e^{-\frac{2\,gx}{V^2}}\right]. \tag{8}$$

Equation (8) expresses the velocity in terms of the initial velocity, the terminal velocity, and the distance fallen. As the distance fallen increases, the velocity approaches the terminal velocity.

PROBLEMS

1. A body weighing 20 lb. moves in a straight line on a rough horizontal plane ($\mu = 0.2$) under the action of a force $F = (t^3 + 12\,t + 4)$ lb. If the body starts from rest, find the velocity and distance after 4 sec.
Ans. 257.6 ft. per second, 288.5 ft.

2. If a body falls from an infinite distance to the surface of the earth under an attraction that varies inversely as the square of the distance, with what velocity will it reach the earth, assuming the earth to be fixed in space? *Ans.* $\sqrt{2\,gR}$.

3. What is the least velocity with which a body must be projected perpendicularly from the surface of the moon in order that it may never return? The radius of the moon is one fourth, and its mass is one eighty-first, that of the earth. *Ans.* 1.54 mi. per second.

4. A weight of 5 lb. having an initial velocity of 100 ft. per second moves in a medium whose resistance is numerically $\frac{\sqrt{v}}{g}$ lb. Find the time required for the weight to come to rest. *Ans.* 100 sec.

5. A 500-pound falling bomb has a terminal velocity in air of 800 ft. per second; find the value of the proportionality factor k if the air resistance is kv^2. *Ans.* $\frac{1}{1280}$.

6. Find the time from rest required for the falling bomb of Problem 5 to attain a speed of 700 ft. per second. *Ans.* 33.6 sec.

7. Find the distance that the bomb in Problem 6 falls.
Ans. 14,420 ft.

8. Assuming that the air resistance to the motion of a sphere is proportional to its area, show that the terminal velocities of spheres are to each other as the square roots of their radii.

146. Simple harmonic motion. If a particle moves on a straight line under the action of a force which is always directed toward a fixed point O on the straight line and the magnitude of the force is proportional to the displacement x of the particle from the point O, the motion of the particle is called simple harmonic. This type of motion is associated with many physical phenomena; for example, water and sound waves, the simple pendulum, the rolling of ships, elastic vibrations in shafts and machine parts, the most general case of a dynamical system having one degree of freedom and oscillating about a position of stable equilibrium, etc.

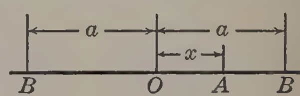

FIG. 277

Let k denote the magnitude of the force at unit distance from O; then $-kx$ denotes the force at a distance x. The negative sign is placed before kx since the acceleration $\frac{d^2x}{dt^2}$ is positive in the direction in which x increases, that is, opposite in direction to the force. The equation of simple harmonic motion is

$$m\frac{d^2x}{dt^2} = -kx. \tag{1}$$

Multiplying each member of the equation by $2\frac{dx}{dt}$ and integrating,

$$m\left(\frac{dx}{dt}\right)^2 = -kx^2 + c. \tag{2}$$

Let $OB = a$ be the value of x when the velocity $\frac{dx}{dt} = 0$; then (2) becomes

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{k}{m}}\,(a^2 - x^2). \tag{3}$$

Equation (3) expresses the velocity of the particle in terms of the displacement x.

Separating the variables,

$$\pm\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\frac{k}{m}} \cdot dt. \tag{4}$$

Integrating again,

$$\text{arc sin}\,\frac{x}{a} = \sqrt{\frac{k}{m}}\,t + c_1. \tag{5}$$

Assuming that the particle is at the fixed point O when $t=0$, (5) becomes

$$\text{arc sin}\,(0) = c_1.$$

Hence $c_1 = 0, \pi, 2\pi, \cdots, n\pi$, where n is any integer, and, taking the principal value only, (5) reduces to

$$\text{arc sin}\,\frac{x}{a} = \sqrt{\frac{k}{m}}\,t, \tag{6}$$

or

$$x = a \sin\sqrt{\frac{k}{m}}\,t. \tag{7}$$

As t increases, the sine oscillates between $+1$ and -1, and therefore x oscillates between $+a$ and $-a$. When the particle has its maximum displacement a, called its *amplitude*, (7) gives

$$a = a \sin\sqrt{\frac{k}{m}}\,t,$$

and hence

$$\sqrt{\frac{k}{m}}\,t = \text{arc sin}\,1 = \frac{\pi}{2}, \tag{8}$$

or

$$t = \sqrt{\frac{m}{k}} \cdot \frac{\pi}{2}.$$

The *period* T of the motion, or the time required for the particle to make one complete oscillation, is $4t$. Hence

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{9}$$

The velocity of the particle may also be expressed in terms of t by differentiating (7), which gives

$$v = \frac{dx}{dt} = a\sqrt{\frac{k}{m}} \cos\sqrt{\frac{k}{m}}\,t. \tag{10}$$

EXAMPLES

1. A body weighing 10 lb. is attached to the free end of a vertical spring having a modulus of 2 lb. per inch. Find its motion if the body is pulled down 2 in. below its equilibrium position and released.

Solution. Let x be the extension of the spring measured positively downward from the equilibrium position. Then the force acting on the body is $10 - 2(5 + x) = -2x$ lb. when x is measured in inches, or $-24x$ lb. when x is measured in feet. Hence the equation of motion is

$$\frac{10}{g}\frac{d^2x}{dt^2} = -24x.$$

The motion of the body is therefore simple harmonic. The period is found from the equation

$$T = 2\pi\sqrt{\frac{m}{k}},$$

where m is the mass of the body and k is the force at unit distance from the equilibrium position. Since $m = \frac{W}{g}$ and since $g = 32.2$ *feet* per second per second, the constant k must also be expressed in pounds per *foot*. Therefore

$$T = 2\pi\sqrt{\frac{10}{32.2}\cdot\frac{1}{24}} = 0.72 \text{ sec.}$$

The amplitude is $\frac{1}{6}$ ft. The maximum velocity may be found from the equation

$$v = \sqrt{\frac{k}{m}}\sqrt{a^2 - x^2},$$

by placing $x = 0$, which gives the maximum velocity

$$v = 1.47 \text{ ft. per second.}$$

2. A weight W resting on a smooth table is attached to the middle point of a stretched spring whose modulus is k. Determine the motion if the weight is displaced along the spring and released.

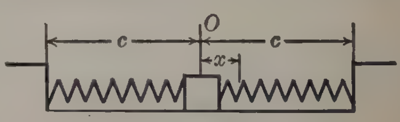

FIG. 278

Solution. Let $2d$ be the natural length of the spring and $2c$ its stretched length. If the weight is displaced a distance x to the right of O, the tension in the left half of the spring is $k(c - d + x)$ and in the right half $k(c - d - x)$. The resultant force acting on the weight is the difference of these tensions, which is equal to $2kx$, and is directed toward O. Hence the equation of motion is

$$\frac{W}{g}\frac{d^2x}{dt^2} = -2kx,$$

and the motion is simple harmonic. The period of the motion is

$$T = 2\pi\sqrt{\frac{W}{2gk}},$$

which is independent of c and d.

PROBLEMS

1. The maximum velocity of a particle moving with simple harmonic motion is 4 ft. per second and its period is $\frac{1}{2}$ sec. Find its amplitude and maximum acceleration.

Ans. $\dfrac{1}{\pi}$ ft., $16\,\pi$ ft. per second per second.

2. The amplitude of a simple harmonic motion is 2 ft. and the period of the motion is 1 sec. Find the velocity and acceleration of the particle when it is 1 ft. from its equilibrium position.

Ans. $2\,\pi\sqrt{3}$, $4\,\pi^2$.

3. A weight of 8 lb. is attached to a spring whose modulus is 2 lb. per inch. The weight is released when the spring is unstressed. Find the distance it falls before coming to rest and the period of its motion. Also find the maximum velocity of the weight. Neglect the mass of the spring. *Ans.* 8 in., 0.639 sec., 3.28 ft. per second.

4. A 10-pound weight elongates a spring 1 ft. If in its position of equilibrium it is given a downward velocity of 1 ft. per second, how far will it move down before starting back? *Ans.* 0.176 ft.

5. When a load is applied suddenly to any yielding structure the resulting stresses are greater than would result from the gradual application of the same load. How much greater? *Ans.* Double.

6. A 5-pound weight falls from a height of 4 in. upon a spring whose modulus is 2.5 lb. per inch. How far will the spring be depressed, and what time elapses during the depression?

Ans. 0.54 ft., 0.146 sec.

7. A 25-ton car moving 1 mi. per hour strikes a bumping post. The post and the draft gear combined offer a resistance which follows Hooke's law, the total displacement being 5 in. Find the maximum acceleration of the car. Will the baggage in the car shift if the coefficient of friction is 0.4? *Ans.* 5.16 ft. per second per second.

8. Prove that if a straight tunnel were bored from Minneapolis to Chicago, a train would traverse it under the force of gravity alone in about 42 min., friction being neglected and the attraction being assumed proportional to the distance from the center of the earth. How long would it take a ball to fall through a tunnel from the north pole to the south pole?

9. A weight of 32.2 lb. is suspended from a spiral spring. If the period of vibration is 0.314 sec., find the modulus of the spring.

Ans. 33.3 lb. per inch.

10. A cube whose edge is 1 ft. and whose specific gravity is 0.5 floats in water in an upright position. Find the time of vertical oscillation if it is slightly depressed and then released. *Ans.* $t = 0.783$ sec.

CHAPTER XV

CURVILINEAR PLANE MOTION

147. Equations of motion. Let the motion of a particle of mass m be referred to rectangular axes in its plane of motion, and let X and Y be the x and y components of the forces acting upon the particle. Since each force produces an effect which is independent of the other forces acting, the equations of motion are

$$X = m\frac{d^2x}{dt^2}, \tag{1}$$

$$Y = m\frac{d^2y}{dt^2}. \tag{2}$$

The path of the particle in the plane is found by solving the differential equations (1) and (2) for x and y. If x and y are determined in terms of a common variable t, the equation of the path is determined by eliminating t between the expressions for x and y.

148. Motion of a particle under gravity, air resistance being neglected. Let the x axis be taken along the horizontal, and let the y axis be taken vertical and positive upward. If W is the weight of the particle, the equations of motion are

$$\frac{W}{g}\frac{d^2x}{dt^2} = 0, \tag{1}$$

$$\frac{W}{g}\frac{d^2y}{dt^2} = -W. \tag{2}$$

Integrating with respect to the time,

$$\frac{dx}{dt} = c_1, \tag{3}$$

$$x = c_1t + c_2; \tag{4}$$

$$\frac{dy}{dt} = -gt + c_3, \tag{5}$$

$$y = -\frac{gt^2}{2} + c_3t + c_4. \tag{6}$$

Let V_0 be the initial velocity of projection of the particle, and let θ be the angle which the initial velocity makes with the x axis. The vertical component of the initial velocity is $V_0 \sin \theta$, and the horizontal component is $V_0 \cos \theta$. The initial conditions of the motion are that $x = 0$ and $y = 0$ when $t = 0$; and $\frac{dx}{dt} = V_0 \cos \theta$ and $\frac{dy}{dt} = V_0 \sin \theta$ when $t = 0$. Making use of the initial conditions to determine the constants, (3), (4), (5), and (6) become

$$\frac{dx}{dt} = V_0 \cos \theta, \tag{7}$$

$$x = V_0 t \cos \theta; \tag{8}$$

$$\frac{dy}{dt} = - gt + V_0 \sin \theta, \tag{9}$$

$$y = - \frac{gt^2}{2} + V_0 t \sin \theta. \tag{10}$$

Equations (7), (8), (9), and (10) give the coördinates and the components of the velocity after any time t in terms of the initial velocity and the angle of projection. The elimination of t between (8) and (10) gives the equation of the path,

$$y = - \frac{gx^2}{2\,V_0^2} \sec^2 \theta + x \tan \theta. \tag{11}$$

If the particle is projected below the x axis, the angle θ is negative and hence $V_0 \sin \theta$ is negative.

A useful artifice in working problems of this type is to imagine two particles projected at the same instant, one along the y axis, having an initial velocity of $V_0 \sin \theta$, and the second along the x axis, with a velocity of $V_0 \cos \theta$. The velocity and position of the first may be found at any instant by the equations of vertical rectilinear motion under gravity (§ 143), while the velocity of the second remains constant. Two lines drawn through the imaginary particles parallel to the axes serve to locate by their intersection the position of the actual particle.

PROBLEMS

1. Show that the equation of the path of a projectile in vacuo is the equation of a parabola, concave downward, having its vertex at the point

$$\left(x = \frac{V_0^2 \sin \theta \cos \theta}{g}, \quad y = \frac{V_0^2 \sin^2 \theta}{2\,g}\right).$$

Find the equation of the path when the curve is referred to parallel axes at the vertex.

2. Employ the artifice of § 148 to show that the maximum height of the projectile above the point of projection is $d = \frac{V_0^2 \sin^2 \theta}{2\,g}$. Also make use of the same method to show that the range on a level plane drawn through the point of projection is $\frac{V_0^2 \sin 2\,\theta}{g}$.

3. A gun fires a projectile with a velocity of 1000 ft. per second at an angle of 30° above the horizontal. Find the maximum height attained, the total time of flight, and the range.

Ans. 3882 ft., 31.06 sec., 26,890 ft.

4. A projectile has a range of 500 ft. and the time of flight is 5 sec. Find the velocity of projection and the angle of elevation of the gun. *Ans.* 128 ft. per second, $\theta = \tan^{-1} 0.805$.

5. A projectile is fired from a position 1610 ft. above a level plane. The muzzle velocity is 644 ft. per second, and the inclination of the gun to the horizontal is 30°. Find the range on the level plane.

Ans. 13,465 ft.

6. A projectile fired from the origin of coördinates is observed to pass through the two points (x_1, y_1) and (x_2, y_2). Find the angle of projection. *Ans.* $\tan\theta = \frac{y_2 x_1{}^2 - y_1 x_2{}^2}{x_1 x_2 (x_1 - x_2)}$.

149. Motion in a circle; uniform speed. Let a particle of mass $\frac{W}{g}$ move in a circle of radius r with uniform speed v. The acceleration along the tangent to the circle is zero, since the particle moves with uniform speed. The particle has an acceleration toward the center of the circle expressed by $\frac{v^2}{r}$ or $\omega^2 r$, where ω is the angular velocity. The force required to produce this acceleration must be in the direction of the acceleration (that is, along the radius toward the center of the circle), and its magnitude is

$$\frac{Wv^2}{gr}, \quad \text{or} \quad \frac{W}{g}\omega^2 r.$$

This force is sometimes called the *centripetal force.*

The equal and opposite reaction which the particle exerts upon the agency which causes the particle to move in a circle is called *centrifugal force.*

PROBLEMS

1. A particle of weight W moves in a circle of radius r with a uniform angular velocity of n revolutions per minute. Find the centrifugal force. *Ans.* $\dfrac{W\pi^2n^2r}{900\,g}$.

2. A man weighing 161 lb. rides upon a merry-go-round which turns uniformly 6 times per minute. Find the inward force which the machine exerts upon the man if his radial distance from the center is 15 ft. *Ans.* 29.6 lb.

150. The conical pendulum. Let a particle of weight W be attached to a fixed point A by means of a light string of length l. Let it be required to find the time of revolution if the particle describes a horizontal circle of radius $l \sin \theta$, where θ is the angle which the string makes with the vertical. Let P be the tension in the string, and let the angular velocity be ω. Resolving horizontally,

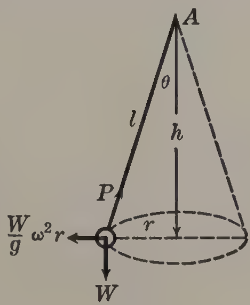

FIG. 279

$$P \sin \theta = \frac{W}{g}\, \omega^2 l \sin \theta, \tag{1}$$

and resolving vertically,

$$P \cos \theta = W. \tag{2}$$

Eliminating P from (1) and (2),

$$\omega^2 = \frac{g}{l \cos \theta} = \frac{g}{h}, \tag{3}$$

where h is the height of the cone.

The time for a single revolution is the total angle described divided by the angular velocity, or

$$T = \frac{2\,\pi}{\omega} = 2\,\pi\sqrt{\frac{h}{g}}. \tag{4}$$

PROBLEMS

1. The length of the string of a conical pendulum is 4 ft., and the angle θ which the string makes with the vertical is 30°. Find the time of one revolution and the tension in the string if the particle weighs 20 lb. *Ans.* 2.06 sec., 23.1 lb.

2. Find the minimum angular velocity which a simple pendulum l ft. long must have in order that it may behave as a conical pendulum. *Ans.* $\omega = \sqrt{\dfrac{g}{l}}$.

151. Superelevation of railway track. If a locomotive runs on a circular track of radius r with uniform speed v, the forces which act upon it are the same as those which act upon the conical pendulum if the force in the string is replaced by the reaction of the rails and if the track is banked at the proper angle θ to prevent side thrust of the wheels on the rails. If P represents the resultant upward thrust of the rails on the engine,

$$P \sin \theta = \frac{Wv^2}{gr}. \tag{1}$$

Also, from § 150,

$$P \cos \theta = W. \tag{2}$$

Eliminating P,

$$\tan \theta = \frac{v^2}{gr}. \tag{3}$$

If the distance between the rails is G ft., the elevation of the outer rail above the inner rail to prevent side thrust is $G \sin \theta$. If the angle θ is small, the tangent and the sine are nearly equal, and hence the superelevation required at speed v, radius r, and gauge G is

$$G \sin \theta = \frac{v^2 G}{rg}. \tag{4}$$

If a train runs over track at a higher speed than that for which the track is superelevated, the side thrust on the rails is toward the outside of the curve.

PROBLEMS

1. Determine the superelevation of a railway track so that there shall be no flange pressure at a speed of 45 mi. per hour. The distance between the rails is 4 ft. $8\frac{1}{2}$ in., and $r = 1000$ ft. *Ans.* 7.7 in.

2. A locomotive weighing 150 T. rests on a track of radius 2000 ft. Find the total side thrust on the rails if the superelevation is such as to give no side thrust at a speed of 30 mi. per hour. What will be the side thrust at 60 mi. per hour? *Ans.* 9060 lb., 27,180 lb.

3. The maximum speed on a curve of radius 2000 ft. is 60 mi. per hour, and the minimum speed is 15 mi. per hour. What must be the angle between the ties and the horizontal so that the outward thrust of the fastest train shall just equal the inward thrust of the slowest train? *Ans.* $\tan \theta = 0.064$.

4. An automobile track is banked at 45° on a curve of radius 200 ft. If the coefficient of friction between the wheels and the track is 0.3, find the maximum speed on the track when skidding impends. Also find the minimum speed.

Ans. 109.3 ft. per second, 58.9 ft. per second.

5. A smooth straight wire upon which a small ring is free to slide makes an angle of 30° with a vertical axis about which it rotates with an angular velocity of 30 R. P. M. Find the horizontal distance from the axis to the ring in its position of equilibrium. *Ans.* 5.65 ft.

6. A smooth curved rod AB turns about the vertical axis AC with angular velocity ω. Determine the nature of the curve so that a smooth ring D shall remain at rest upon the curved rod in any position.

$$Ans.\ x^2 = \frac{2gy}{\omega^2}.$$

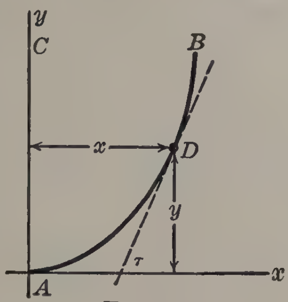

FIG. 280

152. The simple pendulum. A simple pendulum consists of a heavy particle swinging in a vertical plane at the end of a string or rod of negligible weight. Let W be the weight of the particle and θ the angle which the string of length l makes with the vertical. The forces acting on the particle are the weight W and the tension of the string, friction being neglected. Since the tension in the string acts normal to the path, the resultant force along the tangent to the path is $W \sin \theta$. Equating this force to the product of the mass of the particle and its acceleration,

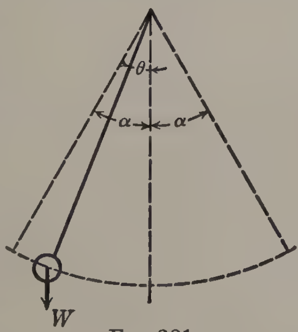

FIG. 281

$$W \sin \theta = -\frac{W}{g}\frac{d^2s}{dt^2} = -\frac{Wl}{g}\frac{d^2\theta}{dt^2}. \tag{1}$$

If θ is small, (1) may be written

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta, \tag{2}$$

since the sine of the angle is nearly equal to the angle measured in radians. This type of differential equation has already been considered (§ 146). The time for a complete oscillation may be written at once,

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{3}$$

A more nearly exact solution for larger angles α is given by

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 \sin^2\frac{\alpha}{2} + \left(\frac{1.3}{2.4}\right)^2 \sin^4\frac{\alpha}{2} + \cdots\right]. \tag{4}$$

PROBLEMS

1. An elevator car hangs on a cable 322 ft. long. Find the period of a small oscillation. *Ans.* 19.87 sec.

2. How does the period of a simple pendulum vary with its length?

3. A simple pendulum beats seconds when it swings through 1° on each side of the vertical. Find the number of seconds it will lose per day if it swings through 4°. *Ans.* 24.65 sec.

4. A particle is attached by a string to a fixed point on the surface of a smooth plane inclined at 30° to the horizontal. When the particle is left to itself on the plane its period of oscillation is one second. Find the length of the string. *Ans.* 4.9 in.

153. The energy equation. If the equations of motion,

$$X = m\frac{d^2x}{dt^2} \tag{1}$$

and

$$Y = m\frac{d^2y}{dt^2}, \tag{2}$$

are multiplied by dx and dy respectively, and the results added, it follows that

$$m\left(\frac{d^2x}{dt^2}\,dx + \frac{d^2y}{dt^2}\,dy\right) = X\,dx + Y\,dy. \tag{3}$$

Also, if v is the velocity of the particle in its path,

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

Differentiating with respect to t,

$$\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt} = v\frac{dv}{dt},$$

and multiplying by $m\,dt$,

$$m\left(\frac{d^2x}{dt^2}\,dx + \frac{d^2y}{dt^2}\,dy\right) = mv\,dv. \tag{4}$$

From (3) and (4), $\quad mv\,dv = X\,dx + Y\,dy.$

Integrating between any two points on the path,

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = \int_{(x_0,\,y_0)}^{(x,\,y)} (X\,dx + Y\,dy). \tag{5}$$

Equation (5) states that the change in kinetic energy of a particle is equal to the work done by the external forces on the particle during its motion from one point to another. If a force function exists, the total work depends only upon the initial position and the final position of the particle (§ 105).

154. The motion of a particle in a vertical circle. A particle of weight W attached to a rod of negligible weight describes a circle in a vertical plane. It is required to find the velocity at any point A and the axial stress in the rod.

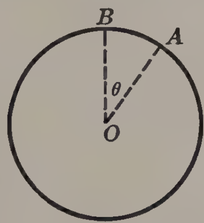

FIG. 282

Let the velocity of the particle at the highest point B of the circle be v_0, and let the angle $AOB = \theta$. The kinetic energy of the particle at B is $\frac{1}{2}mv_0^2$, and, since the system is conservative, the work done by gravity as the particle moves from B to A is equal to $Wr(1 - \cos\theta)$. Therefore, by the energy equation,

$$\frac{Wv^2}{2g} = \frac{Wv_0^2}{2g} + Wr(1 - \cos\theta), \tag{1}$$

where v is the velocity at the point A.

Also, if T is the thrust in the rod,

$$W\cos\theta - T = \frac{Wv^2}{gr},$$

from which

$$T = W\left(\cos\theta - \frac{v^2}{gr}\right). \tag{2}$$

EXAMPLES

1. A particle of weight W describes a vertical circle under the action of gravity and the tension in a string joining the particle to the center. Find the least velocity at the lowest point of the path so that the particle may describe a complete circle.

Solution. Since a string cannot resist a thrust, the value of $W\cos\theta$ must not be larger than $\frac{Wv^2}{gr}$ (§ 154). But $W\cos\theta$ is a maximum when $\theta = 0°$, or $\frac{Wv^2}{gr} = W$, from which $v^2 = gr$. Hence the velocity at the highest point of the circle must be at least equal to $\sqrt{gr}$. Substituting $v_0 = \sqrt{gr}$ in (1), § 154, and letting $\theta = \pi$, the minimum value which v may have in order that the particle may describe a circle is given by $v^2 = 5\,gr$.

2. A particle of weight W at rest on the top of a smooth fixed sphere is slightly displaced from its equilibrium position. Find the angle θ which a radius through the particle makes with the vertical when the particle leaves the surface of the sphere.

Solution. The particle will leave the sphere when the normal pressure between the particle and the sphere changes sign. Hence, by (2), § 154,

$$\cos\theta = \frac{v^2}{gr}.$$

From (1), § 154, $\quad v^2 = 2\,gr(1 - \cos\theta),$

since $v_0 = 0$. Eliminating v, $\quad \cos\theta = \frac{2}{3}.$

Hence the particle leaves the sphere and describes a parabola when $\cos\theta = \frac{2}{3}$.

PROBLEMS

1. A weight of 322 lb. is suspended from a rope 10 ft. long. The weight is drawn away from the vertical through an angle of 45° and let go. Find the tension in the rope when it swings through the vertical. *Ans.* 510.7 lb.

2. An airplane passes through the lowest point of a vertical curve at a speed of 150 ft. per second. If the radius of curvature is 150 ft. and the weight of the aviator is 140 lb., find the pressure between the aviator and the seat of the airplane. *Ans.* 792 lb.

3. A particle starts from rest at the highest point of a smooth sphere 12 ft. in diameter, the center of which is 16 ft. above a horizontal plane. Find the distance the particle slides on the sphere and also the distance between the point where the vertical diameter of the sphere pierces the plane and the point where the particle strikes the plane. Also find the velocity and angle at which the particle strikes the plane. Determine the equation of the path of the particle referred to vertical and horizontal axes at the point of departure from the sphere, the y axis being taken positive downward.

Ans. 5.04 ft., 11.15 ft., $V = 37.6$ ft. per second,

$$\tan\theta = 4.87,\ y = \frac{9}{32}x^2 + \frac{\sqrt{5}}{2}x.$$

4. A heavy particle is suspended from a fixed point by a string 2 ft. long. The initial horizontal velocity of projection when the particle is in its lowest position is 16.04 ft. per second. Find the vertical distance of the particle above its lowest point when the string becomes slack. *Ans.* 3.33 ft.

5. A rod 8 ft. long, of negligible weight, is bent at its middle into the form of a right angle, and two particles weighing 8 lb. each are attached to its ends. The rod rotates in a vertical plane about a

horizontal axis at its center. The initial velocity of one of the weights when it is vertically above the axis is 16 ft. per second. Find its velocity after it has turned through an angle θ from the vertical, and find the maximum velocity of the weights.

Ans. $V = 2\sqrt{64 + g(1 + \sin\theta - \cos\theta)}$,
$V_{\text{max.}} = 23.8$ ft. per second.

6. Find the height above the plane CD, Fig. 283, where a particle must be placed on the incline so that it will pass the highest point A of the loop without losing contact. If the particle is placed at a height of 50 ft. above the plane CD, find the distance x to the point where it strikes the plane. *Ans.* 35 ft., 33.2 ft.

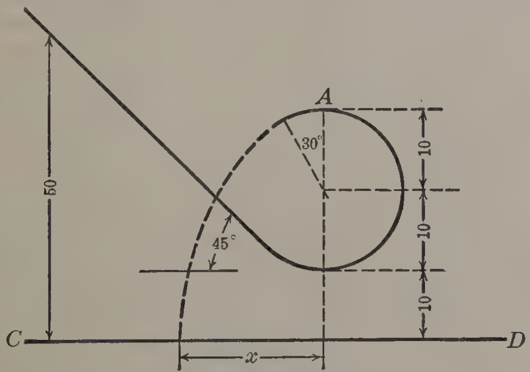

FIG. 283

155. Motion on a smooth fixed curve under gravity. The velocity of a particle moving on a smooth fixed curve in a vertical plane may be obtained at any point on the curve by using the energy equation. Since there is no friction, the work of friction is zero. It follows, therefore, that if the particle moves from one position on the curve to a second position, the difference in kinetic energy in the two positions is just equal to the weight of the particle multiplied by the vertical distance between the two positions. Usually it is not easy to obtain the time required for the particle to pass from one position to the other.

PROBLEMS

1. A particle slides down a smooth wire bent in the form of a quadrant of an ellipse having the major axis vertical. What will be its velocity at the lower extremity of the major axis if it starts from rest at the extremity of the minor axis? The major and minor axes are 32.2 ft. and 16.1 ft. long respectively. *Ans.* 32.2 ft. per second.

2. The particle of Problem 1 leaves the wire at the lower extremity of the major axis and falls to a platform 64.4 ft. below. Find the velocity of the particle when it strikes the platform.

Ans. $V = 32.2\sqrt{5}$ ft. per second, $\tan\theta = 2$.

3. A particle moves on a plane curve under the action of no force. Show that its speed is constant and that the pressure of the particle on the curve is proportional to the curvature.

4. A particle of weight W moves on a smooth fixed curve AB lying in a vertical plane under the action of gravity, from a point (x_0, y_0) to a point (x, y). Show that

$$\tfrac{1}{2}\,mv^2 - \tfrac{1}{2}\,mv_0{}^2 = -W(y - y_0)$$

and
$$\frac{Wv^2}{gR} = -W\cos\tau + P,$$

where τ and R are the inclination of the tangent and the radius of curvature at the point (x, y) and P is the pressure of the curve on the particle.

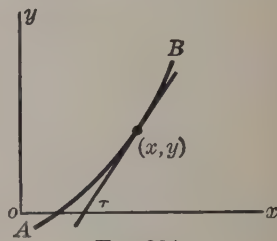

FIG. 284

5. A small bead of weight W moves on a smooth wire in the form of a circle lying in a vertical plane. Show that the sum of the pressures on the particle when it passes through the extremities of any diameter is constant.

156. Motion on a revolving curve. A smooth curve AB rotates in its own plane about a fixed point o with angular velocity ω and angular acceleration α. Find the equations of motion of a particle C of mass m which is constrained to move on the curve with relative velocity u under the action of external forces whose radial and transverse components are P and Q.

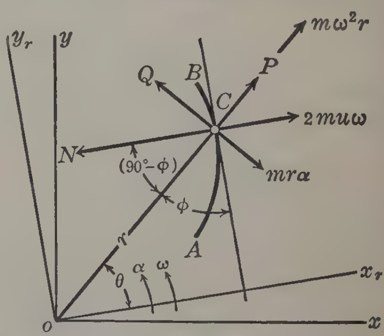

FIG. 285

Let (r, θ) be the polar coördinates of the particle referred to axes $x_r o y_r$, which rotate with the moving curve, and let ϕ be the angle between the tangent to the curve and the radius vector r. Also let xoy be a set of axes fixed in space, with reference to which the motion is required.

Then, by the theorem of Coriolis (§ 141), the equations of motion are

$$P - N\sin\phi = m\Big[\overbrace{-\,\omega^2 r}^{\text{ent.}} + \overbrace{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2}^{\text{rel.}} \overbrace{-\,2\,u\omega\sin\phi}^{\text{comp.}}\Big] \qquad (1)$$

$$Q + N\cos\phi = m\left[\,+\,r\alpha + 2\,\frac{dr}{dt}\frac{d\theta}{dt} + r\,\frac{d^2\theta}{dt^2} + 2\,u\omega\cos\phi\right], \qquad (2)$$

where N is the pressure between the moving particle and the curve.

Resolving the relative velocity u of the particle along and perpendicular to the radius vector,

$$r\,\frac{d\theta}{dt} = u\sin\phi \quad\text{and}\quad \frac{dr}{dt} = u\cos\phi. \qquad (3)$$

Equations (1) and (2) are the equations which would have been obtained for the relative motion with respect to the curve if the curve had been fixed in space and if two fictitious forces,

$$+\,m\omega^2 r \quad\text{and}\quad -\,mr\alpha,$$

had acted upon the particle in addition to the external forces, and if the reaction N of the curve had been replaced by $N - 2\,mu\omega$. The direction of $m\omega^2 r$ and of $-\,mr\alpha$ is opposite to the usual direction; that is, $m\omega^2 r$ acts away from the center of rotation and $-\,mr\alpha$ acts opposite to the direction of $r\alpha$.

The relative velocity u along the curve may be obtained by resolving the impressed and fictitious forces along the tangent. Remembering that $u\,\dfrac{du}{ds} = a$,

$$ma = mu\,\frac{du}{ds} = P\cos\phi + Q\sin\phi + m\omega^2 r\cos\phi - mr\alpha\sin\phi. \qquad (4)$$

Replacing $\sin\phi$ and $\cos\phi$ by $r\,\dfrac{d\theta}{ds}$ and $\dfrac{dr}{ds}$ respectively, (4) becomes

$$mu\,du = (P + m\omega^2 r)dr + (Q - mr\alpha)r\,d\theta. \qquad (5)$$

The reaction N between the particle and the curve may be obtained by resolving the impressed and fictitious forces along the normal to give

$$\frac{mu^2}{R} = -P\sin\phi + Q\cos\phi - m\omega^2 r\sin\phi - mr\alpha\cos\phi + N - 2mu\omega, \qquad (6)$$

where R is the radius of curvature.

Replacing $\sin\phi$ and $\cos\phi$ as before,

$$\frac{mu^2}{R} = -\,(P + m\omega^2 r)r\,d\theta + (Q - mr\alpha)dr + N - 2\,mu\omega. \qquad (7)$$

EXAMPLE

A smooth straight tube rotates in a horizontal plane about an axis through one end with constant angular velocity ω. Investigate the motion of a small sphere of mass m within the tube.

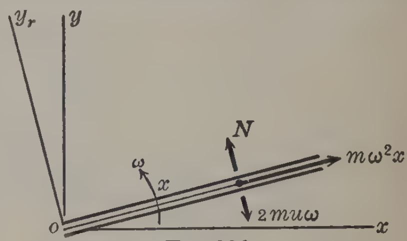

FIG. 286

Solution. The relative motion of the sphere with respect to the tube is the same as if the tube were at rest and the sphere were subjected to (*a*) the actual reaction N of the tube, (*b*) the fictitious force $mx\omega^2$ directed away from the axis of rotation o, (*c*) the fictitious force $2\,mu\omega$ directed perpendicular to the tube and opposite in direction to the complementary acceleration of Coriolis.

The equation of relative motion is, therefore,

$$m\frac{d^2x}{dt^2} = m\omega^2 x$$

or

$$\frac{d^2x}{dt^2} = \omega^2 x, \tag{1}$$

where, for simplicity, x is used instead of x_r.

The solution of this differential equation is

$$x = Ae^{\omega t} + Be^{-\omega t}, \tag{2}$$

where A and B are constants depending on the initial conditions.

It may easily be verified that this is the solution by simple differentiation, as follows:

$$\frac{dx}{dt} = \omega(Ae^{\omega t} - Be^{-\omega t}), \tag{3}$$

$$\frac{d^2x}{dt^2} = \omega^2(Ae^{\omega t} + Be^{-\omega t}) = \omega^2 x. \tag{4}$$

If it is assumed that $x = x_0$ and $\dfrac{dx}{dt} = u_0$ when $t = 0$, then (2) and (3) give

$$x_0 = A + B \tag{5}$$

and

$$u_0 = \omega(A - B). \tag{6}$$

From (5) and (6),

$$A = \frac{\omega x_0 + u_0}{2\,\omega},$$

$$B = \frac{\omega x_0 - u_0}{2\,\omega},$$

and from (2),

$$x = \frac{1}{2\,\omega}[(\omega x_0 + u_0)e^{\omega t} + (\omega x_0 - u_0)e^{-\omega t}]. \tag{7}$$

Also, from (3),

$$u = \tfrac{1}{2}[(\omega x_0 + u_0)e^{\omega t} - (\omega x_0 - u_0)e^{-\omega t}]. \tag{8}$$

The relation between u and x may be obtained by eliminating t between (7) and (8). It may be obtained more simply, however, by multiplying both members of (1) by $2\frac{dx}{dt}$ and integrating. Hence

$$u^2 = \omega^2 x^2 + c, \tag{9}$$

and, from the initial conditions, $c = u_0^2 - \omega^2 x_0^2$.

Therefore

$$u^2 - u_0^2 = \omega^2(x^2 - x_0^2), \tag{10}$$

which gives the relative velocity in terms of the relative displacement.

The reaction of the tube N is obtained by resolving the forces perpendicular to the tube. Thus

$$N = 2\,mu\omega. \tag{11}$$

PROBLEMS

1. A centrifugal gun consists of a smooth straight tube 3 ft. long which rotates in a horizontal plane about a vertical axis through one end with an angular velocity of 2400 R.P.M. Bullets weighing 1 oz. are fed into the tube at a point 6 in. from the axis of rotation. Find the relative and absolute velocity of a bullet and the force which the tube exerts upon it just before it emerges from the tube. Also find the time required for the bullet to traverse the tube.

Ans. 743 ft. per second, 1060 ft. per second, 726 lb., 0.00072 sec.

2. A slender tube AB rotates in a horizontal plane about a vertical axis through the point o with a uniform angular velocity ω. A particle of mass m within the tube has a relative velocity u_0 when it is at a distance r_0 from o. Show that the relative motion is determined by the equations

$$m\frac{d^2x}{dt^2} = mr\omega^2\cos\phi = m\omega^2 x, \quad (1)$$

$$u^2 - u_0^2 = \omega^2(r \;\; - r_0^2), \quad (2)$$

and $\quad N = 2\,mu\omega - ma\omega^2. \quad (3)$

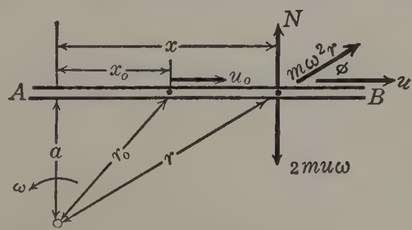

FIG. 287

3. Assume that the tube in Problem 2 is at rest and that it suddenly begins to rotate with an angular velocity ω when the particle is at rest at A. Show that the particle will travel a distance $\frac{a}{2}(e^{\omega t} - e^{-\omega t})$ along the tube in time t and that the pressure of the particle against the tube is then $ma\omega^2(e^{\omega t} + e^{-\omega t} - 1)$.

4. A smooth plane inclined at an angle of 30° to the horizontal is given an acceleration g in a horizontal direction. Show that a particle initially at rest on the plane will move a distance $\frac{gt^2}{4}(\sqrt{3} - 1)$ relative to the plane in time t.

5. A small sphere is at rest at B within a smooth tube bent in the form of a circle of radius r. The tube begins to turn in a horizontal plane about a fixed point A with uniform angular velocity ω. Find the pressure between the sphere and the tube when the sphere passes through the point C.

Ans. $m\omega^2 r(3 - 2\sqrt{2})$.

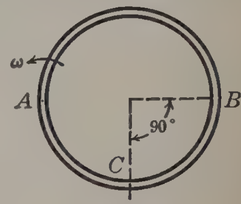

FIG. 288

6. A particle is at rest at the lowest point of a smooth circular tube lying in a vertical plane and having a radius r. Find the maximum height to which the particle will rise when the tube is given a horizontal acceleration g in its plane. Also find the maximum relative velocity and describe the motion of the particle.

Ans. r, $\sqrt{2\,gr(\sqrt{2} - 1)}$.

7. A slender rod rotates with constant angular velocity ω about a horizontal axis through one end. Initially the rod is horizontal and a small bead is placed on the rod at the axis. Show that the bead will exert no pressure on the rod after a time t, where t is given by the relation $4 \cos \omega t = e^{\omega t} + e^{-\omega t}$.

HINT. The equation of motion is

$$mx\omega^2 + mg \sin \omega t = m\frac{d^2x}{dt^2}.$$

CHAPTER XVI

DYNAMICS OF A RIGID BODY

157. The equations of motion. A rigid body has been defined as one in which the distances between the particles constituting the body remain invariable. Although no bodies are absolutely rigid, they may be regarded as rigid in many of the problems of mechanics.

Let a rigid body be divided into n particles, each of mass m, and let (x, y, z) be the coördinates of any particle referred to a system of fixed axes. The forces acting upon any particle may be classified as *external* forces and *internal* forces. If the particle is situated wholly within the interior of the body, the external forces are those which act at a distance, such as gravitational and electrical attractions existing between the particle and *other bodies.* If the particle lies in the surface of the body, the external forces may also include forces due to direct contact with other bodies. The internal forces are those forces which act between the particles of the body.

Let X, Y, and Z be the rectangular components of the resultant of the external forces acting on any particle and P, Q, and R be the components of the resultant of all the internal forces acting on the same particle. Three equations of motion of the particle are

$$X + P = m\frac{d^2x}{dt^2}, \tag{1}$$

$$Y + Q = m\frac{d^2y}{dt^2}, \tag{2}$$

$$Z + R = m\frac{d^2z}{dt^2}. \tag{3}$$

Writing (1) for each of the n particles of the body and adding these equations gives

$$\Sigma X + \Sigma P = \Sigma\left(m\frac{d^2x}{dt^2}\right). \tag{4}$$

Similarly,
$$\Sigma Y + \Sigma Q = \Sigma\left(m\,\frac{d^2y}{dt^2}\right), \tag{5}$$

and
$$\Sigma Z + \Sigma R = \Sigma\left(m\,\frac{d^2z}{dt^2}\right). \tag{6}$$

When the internal forces P for every particle are added together the resultant ΣP becomes zero by virtue of the principle of the equality of action and reaction. In other words, the force that a particle A exerts on a particle B is equal and opposite to the force which B exerts on A, and, since this is true for the forces existing between all the particles, the resultant ΣP vanishes.

Hence (4), (5), and (6) become

$$\Sigma X = \Sigma\left(m\,\frac{d^2x}{dt^2}\right), \tag{7}$$

$$\Sigma Y = \Sigma\left(m\,\frac{d^2y}{dt^2}\right), \tag{8}$$

$$\Sigma Z = \Sigma\left(m\,\frac{d^2z}{dt^2}\right), \tag{9}$$

where ΣX means the sum of the x components of all the *external* forces acting on the body.

Additional equations of motion of the particle are obtained by taking moments of the forces about the axes as in § 55.

Multiplying both sides of (1) by $(-y)$ and both sides of (2) by x and adding,

$$(Yx - Xy) + (Qx - Py) = m\left(x\,\frac{d^2y}{dt^2} - y\,\frac{d^2x}{dt^2}\right). \tag{10}$$

Writing a similar equation for each particle and adding them,

$$\Sigma(Yx - Xy) + \Sigma(Qx - Py) = \Sigma\left[m\left(x\,\frac{d^2y}{dt^2} - y\,\frac{d^2x}{dt^2}\right)\right]. \tag{11}$$

The expression $Qx - Py$ for a single particle is the moment of the *resultant* of P, Q, and R about the z axis. Since the resultant internal forces between any two particles occur in pairs, equal and opposite, the moment of any pair about the z axis is zero. Hence the sum of all the moments of the internal forces about the z axis, namely, $\Sigma(Qx - Py)$, vanishes. Equation (11) therefore reduces to

$$\Sigma(Yx - Xy) = \Sigma\left[m\left(x\,\frac{d^2y}{dt^2} - y\,\frac{d^2x}{dt^2}\right)\right]. \tag{12}$$

Proceeding in a similar manner, equations for moments about the x and y axes may be written.

The three moment equations are

$$L = \Sigma(Zy - Yz) = \Sigma\left[m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right)\right], \tag{13}$$

$$M = \Sigma(Xz - Zx) = \Sigma\left[m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right)\right], \tag{14}$$

$$N = \Sigma(Yx - Xy) = \Sigma\left[m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)\right]. \tag{15}$$

Equations (7), (8), (9), (13), (14), and (15) are the general equations of motion of a rigid body. Further development and modification of these equations are necessary to bring them into more convenient forms.

158. The motion of the center of gravity. Let $(\bar{x}, \bar{y}, \bar{z})$ be the coördinates referred to fixed axes of the center of gravity of any body of mass M. Then, by § 38,

$$\Sigma(mx) = M\bar{x}. \tag{1}$$

Differentiating this equation twice with respect to time,

$$\Sigma\left(m\frac{d^2x}{dt^2}\right) = M\frac{d^2\bar{x}}{dt^2}. \tag{2}$$

Hence (7), (8), and (9) of § 157 may be written

$$\Sigma X = M\frac{d^2\bar{x}}{dt^2}, \tag{3}$$

$$\Sigma Y = M\frac{d^2\bar{y}}{dt^2}, \tag{4}$$

$$\Sigma Z = M\frac{d^2\bar{z}}{dt^2}, \tag{5}$$

where $\frac{d^2\bar{x}}{dt^2}$, $\frac{d^2\bar{y}}{dt^2}$, and $\frac{d^2\bar{z}}{dt^2}$ are the component accelerations of the center of gravity of the body.

Hence (3), (4), and (5) are the same as the equations of the motion of a particle of mass M equal to the mass of the body placed at the center of gravity of the body, and acted upon by forces equal in magnitude and parallel to the external forces acting upon the body. This important principle may be stated as follows:

The motion of the center of gravity of any body acted on by any forces is the same as if all the mass were collected at the center of gravity and all the forces were applied at that point parallel to their original directions.

159. Motion relative to axes through the center of gravity parallel to fixed axes. Let a system of rectangular axes have its origin at the center of gravity of the moving body considered in § 158, and let these axes remain parallel to the original axes of reference. Let (x', y', z') be the coördinates of any particle of the body of mass m referred to the new system of coördinates. Then

$$x = \bar{x} + x', \tag{1}$$
$$y = \bar{y} + y', \tag{2}$$
$$z = \bar{z} + z'. \tag{3}$$

Differentiating these equations twice with respect to time gives

$$\frac{d^2x}{dt^2} = \frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2}, \tag{4}$$
$$\frac{d^2y}{dt^2} = \frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2}, \tag{5}$$
$$\frac{d^2z}{dt^2} = \frac{d^2\bar{z}}{dt^2} + \frac{d^2z'}{dt^2}. \tag{6}$$

Substituting the values of y and z from (2) and (3) and the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$ from (5) and (6) in (13), § 157, gives

$$\begin{aligned}\Sigma[Z(\bar{y} + y') - Y(\bar{z} + z')] &= \Sigma\left[m\left(\bar{y}\frac{d^2\bar{z}}{dt^2} - \bar{z}\frac{d^2\bar{y}}{dt^2}\right)\right] \\ &+ \Sigma\left[m\left(y'\frac{d^2z'}{dt^2} - z'\frac{d^2y'}{dt^2}\right)\right] \\ &+ \Sigma\left[m\left(\bar{y}\frac{d^2z'}{dt^2} + y'\frac{d^2\bar{z}}{dt^2} - \bar{z}\frac{d^2y'}{dt^2} - z'\frac{d^2\bar{y}}{dt^2}\right)\right].\end{aligned} \tag{7}$$

Multiplying (1) by m and extending the summation through the body gives
$$\Sigma(mx) = \Sigma m\bar{x} + \Sigma(mx').$$
But
$$\Sigma(mx) = M\bar{x} \text{ (by § 38).}$$
Also
$$\Sigma m\bar{x} = \bar{x}\Sigma m = M\bar{x}.$$

$$\left.\begin{aligned} &\text{Therefore} \quad \Sigma(mx') = 0 \quad \text{and} \quad \Sigma\left(m\frac{d^2x'}{dt^2}\right) = 0. \\ &\text{Similarly} \quad \Sigma(my') = 0 \quad \text{and} \quad \Sigma\left(m\frac{d^2y'}{dt^2}\right) = 0. \\ &\text{Also} \quad \Sigma(mz') = 0 \quad \text{and} \quad \Sigma\left(m\frac{d^2z'}{dt^2}\right) = 0. \end{aligned}\right\} \tag{8}$$

Therefore (7) becomes

$$\Sigma(Z\bar{y} - Y\bar{z}) + \Sigma(Zy' - Yz') = M\left(\bar{y}\frac{d^2\bar{z}}{dt^2} - \bar{z}\frac{d^2\bar{y}}{dt^2}\right) + \Sigma\left[m\left(y'\frac{d^2z'}{dt^2} - z'\frac{d^2y'}{dt^2}\right)\right]. \tag{9}$$

From (5) and (4), § 158,

$$\Sigma Z\bar{y} = M\bar{y}\frac{d^2\bar{z}}{dt^2},$$

$$\Sigma Y\bar{z} = M\bar{z}\frac{d^2\bar{y}}{dt^2}.$$

Hence
$$\Sigma(Z\bar{y} - Y\bar{z}) = M\left(\bar{y}\frac{d^2\bar{z}}{dt^2} - \bar{z}\frac{d^2\bar{y}}{dt^2}\right). \tag{10}$$

Subtracting (10) from (9) gives

$$\Sigma(Zy' - Yz') = \Sigma\left[m\left(y'\frac{d^2z'}{dt^2} - z'\frac{d^2y'}{dt^2}\right)\right], \tag{11}$$

which is the equation of moments that would have been obtained if the center of gravity had been a fixed point. Therefore, *if any forces act upon a body, the motion of the body relative to axes through its center of gravity parallel to fixed axes is the same as if the center of gravity were fixed and the same forces acted upon the body.*

160. Independence of translation and rotation. Two very important dynamical principles were deduced in §§ 158 and 159. They may be stated as follows:

I. *The motion of the center of gravity of a system acted upon by any forces is the same as if the total mass of the system were concentrated at the center of gravity and all the forces were applied in the same sense at that point parallel to their former lines of action.*

II. *If any forces act upon a body, the motion of the body relative to axes through its center of gravity parallel to fixed axes is the same as if the center of gravity were fixed and the same forces acted upon the body.*

161. D'Alembert's principle. The quantity $m\frac{d^2x}{dt^2}$ is called the x component of the effective force acting upon a particle whose mass is m and whose acceleration in the x direction is $\frac{d^2x}{dt^2}$. From (1), § 157,

$$X + P = m\frac{d^2x}{dt^2}, \tag{1}$$

it is clear that the x component of the effective force is equal to the sum of the x component of the external forces and the x component of the internal forces which act on the particle. If (1) is written in the form

$$X + P - m\frac{d^2x}{dt^2} = 0, \tag{2}$$

it is evident that the x components of the external, internal, and reversed effective forces form a system in equilibrium.

Forming a similar equation for each of the n particles of the body and adding them,

$$\Sigma X + \Sigma P - \Sigma\left(m\frac{d^2x}{dt^2}\right) = 0. \tag{3}$$

But since the internal forces form a system in equilibrium,

$$\Sigma P = 0, \tag{4}$$

and (3) becomes $$\Sigma X - \Sigma\left(m\frac{d^2x}{dt^2}\right) = 0. \tag{5}$$

Similarly $$\Sigma Y - \Sigma\left(m\frac{d^2y}{dt^2}\right) = 0, \tag{6}$$

and $$\Sigma Z - \Sigma\left(m\frac{d^2z}{dt^2}\right) = 0. \tag{7}$$

From (5), (6), and (7) it follows that *the reversed effective forces which act on a body are in equilibrium with the external applied forces*. This is known as d'Alembert's principle.

Similarly, the moments of the external forces which act on a body are in equilibrium with the moments of the reversed effective forces.

EXAMPLES

1. A thin hoop of weight W and radius r spins with uniform angular velocity on a smooth horizontal plane about a vertical axis through its center. Find the tension in the hoop produced by the rotation.

Solution. First method. Let the hoop be cut into two equal parts and replace the hoop tension at A and A by the forces T. By § 160, the motion of the center of gravity of the system consisting of one half of the hoop is the same as if the mass $\frac{W}{2g}$ were concentrated at its center of gravity G and the external forces acting at A were translated to act at G.

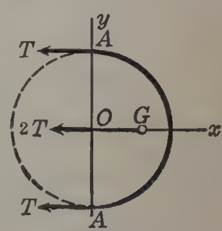

FIG. 289

The distance OG is $\frac{2r}{\pi}$ by § 42. The acceleration of the mass at G is $\omega^2\left(\frac{2r}{\pi}\right)$. Hence the equation of motion is

$$-2T = -\frac{W}{2g}\omega^2\left(\frac{2r}{\pi}\right),$$

from which

$$T = \frac{W\omega^2 r}{2g\pi}.$$

Second method. Let the system consist of an element of the hoop of length $r\,d\theta$ and mass $\frac{W}{g}\cdot\frac{r\,d\theta}{2\pi r}$. Concentrating the mass of the element at its center of gravity and translating the external forces to act at that point, the equation of motion is

$$-2T\sin\left(\frac{d\theta}{2}\right) = -m\omega^2 r = -\frac{W\,d\theta}{2\pi g}\omega^2 r.$$

Replacing $\sin\left(\frac{d\theta}{2}\right)$ by $\frac{d\theta}{2}$ and solving,

$$T = \frac{W\omega^2 r}{2g\pi}.$$

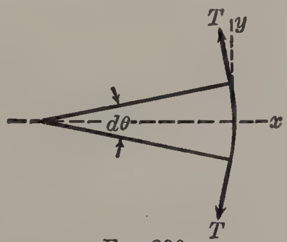

FIG. 290

2. A vertical axis DE supporting a horizontal bar OB, of length a, revolves with uniform angular velocity ω. A uniform bar AB, of length l, hinged to the bar OB at B is inclined at a constant angle θ to the vertical. Find ω.

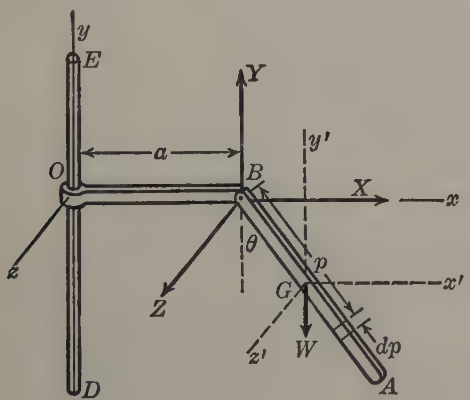

FIG. 291

Solution. First method. The equations of motion of the rod AB referred to the fixed axes are

$$X = \frac{W}{g}\frac{d^2\bar{x}}{dt^2},\quad Y - W = \frac{W}{g}\frac{d^2\bar{y}}{dt^2},\quad Z = \frac{W}{g}\frac{d^2\bar{z}}{dt^2}. \tag{1}$$

The moment equations referred to the axes through the center of gravity are

$$\left.\begin{aligned}\Sigma(Zy' - Yz') &= \Sigma\left[m\left(y'\,\frac{d^2z'}{dt^2} - z'\,\frac{d^2y'}{dt^2}\right)\right],\\ \Sigma(Xz' - Zx') &= \Sigma\left[m\left(z'\,\frac{d^2x'}{dt^2} - x'\,\frac{d^2z'}{dt^2}\right)\right],\\ \Sigma(Yx' - Xy') &= \Sigma\left[m\left(x'\,\frac{d^2y'}{dt^2} - y'\,\frac{d^2x'}{dt^2}\right)\right].\end{aligned}\right\} \qquad (2)$$

The external forces are X, Y, and Z acting at B, and W acting at G. Since the center of gravity G moves in a circle of radius $\left(a + \frac{l}{2}\sin\theta\right)$,

$$\frac{d^2\bar{x}}{dt^2} = -\,\omega^2\left(a + \frac{l}{2}\sin\theta\right),\quad \frac{d^2\bar{y}}{dt^2} = 0,\ \text{ and }\ \frac{d^2\bar{z}}{dt^2} = 0.$$

Hence, from (1),

$$X = -\frac{W\omega^2}{g}\left(a + \frac{l}{2}\sin\theta\right),\quad Y = W,\quad \text{and}\quad Z = 0. \qquad (3)$$

For each element of the rod the component accelerations referred to axes $x'y'z'$ are

$$\frac{d^2x'}{dt^2} = -\,\omega^2 x',\quad \frac{d^2y'}{dt^2} = 0,\quad \text{and}\quad \frac{d^2z'}{dt^2} = 0.$$

The element of the rod at $B(x', y', 0)$ is acted upon by the forces already obtained in (3). Every element of the rod is acted upon by the external force of gravity, and these forces may be replaced by the weight W at G. No other external forces act on any element of the rod. Substituting these values in (2), the first two equations vanish identically. The last equation gives

$$-\frac{Wl}{2}\sin\theta + \frac{W}{g}\,\omega^2\left(a + \frac{l}{2}\sin\theta\right)\frac{l}{2}\cos\theta = \Sigma m[x'(0) - y'(-\,\omega^2 x')]. \qquad (4)$$

But $\Sigma(m\,\omega^2 x'y') = \omega^2\Sigma(mx'y') = \omega^2\int x'y'\,dm = -\,\omega^2\,\frac{Wl^2}{12\,g}\sin\theta\cos\theta,$

from Problem 2, p. 161.

Therefore (4) becomes

$$-\frac{Wl}{2}\sin\theta + \frac{W\omega^2}{g}\left(a + \frac{l}{2}\sin\theta\right)\frac{l}{2}\cos\theta = -\,\omega^2\,\frac{Wl^2}{12\,g}\sin\theta\cos\theta,$$

from which

$$\omega = \sqrt{\frac{3\,g\tan\theta}{3\,a + 2\,l\sin\theta}}.$$

Second method. Select an element of length dp on the rod AB at a distance p from the end B. The mass of the element is $\frac{W}{g}\,\frac{dp}{l}$, and its acceleration is $\omega^2(a + p\sin\theta)$ directed toward the rod DE. Hence the reversed effective force on the element is $\frac{W}{g}\,\frac{\omega^2}{l}\,(a + p\sin\theta)\,dp$ directed away from the rod DE.

The total moment of the reversed effective forces of all the elements about B is

$$\int_0^l \frac{W\omega^2}{gl}(a + p \sin \theta)(p \cos \theta)\, dp,$$

which must equal the moment of the weight of the rod, $\frac{Wl}{2} \sin \theta$. Hence

$$\frac{W\omega^2}{gl}\int_0^l (a + p \sin \theta)(p \cos \theta)\, dp = \frac{Wl}{2} \sin \theta,$$

from which

$$\omega = \sqrt{\frac{3\, g \tan \theta}{3\, a + 2\, l \sin \theta}}.$$

PROBLEMS

1. Let the rod of Example 2, p. 251, be hinged to the axis DE at C. Find the angle θ in terms of l and ω. (Note that if $3\, g > 2\, l\omega^2$, the cosine of θ becomes larger than unity, which is impossible; θ is therefore zero and the rod hangs vertically.)

Ans. $\theta = 0$, or $\cos \theta = \frac{3\, g}{2\, l\omega^2}$.

2. A rough plank 24 ft. long weighing 50 lb. rests on a perfectly smooth horizontal plane and one end of the plank is attached to the free end of a fixed spring scale. A man weighing 150 lb. starts from the end of the plank most remote from the spring scale and runs to the other end, thereby registering a constant force of 30 lb. on the scale. Assuming that the initial tension in the spring is 30 lb., find the time required for the man to run from one end of the plank to the other end. *Ans.* 2.73 sec.

3. If the spring is detached from the plank of Problem 2, find the distance the plank will move when the man walks from one end of it to the other.

Hint. Since there is no external horizontal force acting upon the system comprising the man and the plank, the center of gravity remains at rest.

Ans. 18 ft.

4. If air resistance is neglected, show that the parabolic motion of the center of gravity of a projectile is not changed by the burst of the projectile.

5. Show that if several objects are thrown into the air, the center of gravity of all of them describes a parabola.

6. A rod revolves uniformly about its center on a smooth horizontal plane without translation. If the rod suddenly breaks in two at the center, determine the motion of each part.

7. A uniform rod 6 ft. long weighing 24 lb. rotates about its center of gravity in a horizontal plane with a uniform angular velocity of 360 R.P.M. Find the tension at a point 1 ft. from the center.

Ans. 706.2 lb.

8. Two blocks weighing 4 lb. and 8 lb., respectively, rest on a smooth horizontal plane at the two ends of a compressed spring of negligible mass. Determine the motion of the center of gravity of the blocks if the spring is suddenly allowed to expand.

9. A table having its legs at the vertices of an equilateral triangle whose sides are 1 ft. long stands in a moving railway coach with one side perpendicular to the rails. The center of gravity is in a vertical line through the center of the triangle and 2 ft. above the floor of the coach. Find the pressure on the legs if the table weighs 25 lb. and if the acceleration of the coach is 2 ft. per second per second in the direction in which the triangle points.

Ans. 4.75 lb., 10.12 lb., 10.12 lb.

10. The wheel base of an automobile weighing 4000 lb. is 120 in., and the center of gravity is located at a point 50 in. in front of the rear axle and 24 in. above the ground. If the coefficient of friction between the road and the tires is 0.5, and the speed of the car is 45 mi. per hour, determine the minimum distance in which the car can be stopped (*a*) when only the rear wheels are braked, (*b*) when only the front wheels are braked; (*c*) when all the wheels are braked. Does the result depend upon the weight?

Ans. 255 ft., 292 ft., 135 ft.

CHAPTER XVII

MOTION ABOUT A FIXED AXIS

162. The equation of motion. Let the z axis be taken as the fixed axis, and let the angular displacement of any body about this axis be referred to the xz plane. Since the body is rigid the angular velocity or angular acceleration of all lines in the body which are perpendicular to the axis of rotation is the same at any instant, and therefore the angular velocity or the angular acceleration of any one of these lines serves to specify the angular velocity or the angular acceleration of the body (§ 132). Let r be the perpendicular distance from any particle P of mass m to the axis of rotation, and let θ be the angle which r makes with the xz plane. Since the particle describes a circle about A, its total acceleration may be represented by two components, along PA and perpendicular to PA, given by $r\left(\frac{d\theta}{dt}\right)^2$ and $r\frac{d^2\theta}{dt^2}$ respectively (§ 129). The component forces which must be exerted upon the particle P along PA and perpendicular to PA are therefore $mr\left(\frac{d\theta}{dt}\right)^2$ and $mr\frac{d^2\theta}{dt^2}$, and the moments of these forces about the fixed axis at A are zero and $mr^2\frac{d^2\theta}{dt^2}$ respectively. Since, by the general equation of motion (§ 157), the total moment of the effective forces acting on all the particles, namely, $\Sigma\left(mr^2\frac{d^2\theta}{dt^2}\right)$, must equal the moment N of the impressed forces about the axis of rotation, therefore

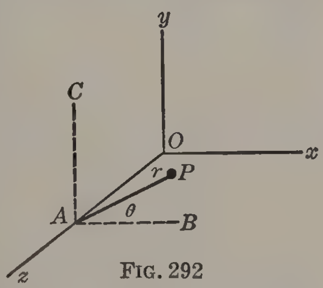

FIG. 292

$$N = \Sigma\left(mr^2\frac{d^2\theta}{dt^2}\right); \tag{1}$$

and since $\Sigma mr^2 = I$ (§ 109), where I is the moment of inertia

of the body about the axis of rotation, (1) may be written

$$N = I\frac{d^2\theta}{dt^2}. \tag{2}$$

If this equation is integrated twice, it gives the value of the angular velocity $\frac{d\theta}{dt}$ and the angle θ in terms of the time t and two constants of integration. The constants may be determined from the initial conditions in the usual way. Equation (2) may be written

$$N = I\alpha, \tag{3}$$

where α is the angular acceleration of the body.

163. Kinetic energy of a body rotating about a fixed axis. The total velocity of the particle P, Fig. 292, § 162, is $r\frac{d\theta}{dt}$ (§ 129). Hence the kinetic energy of the particle is (§ 153)

$$\frac{1}{2}mv^2 = \frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2.$$

The total kinetic energy of the body is therefore

$$\Sigma\left[\frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2\right] = \frac{1}{2}\Sigma\left[mr^2\left(\frac{d\theta}{dt}\right)^2\right] = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2.$$

If ω is written for the angular velocity $\frac{d\theta}{dt}$, the expression for the total kinetic energy becomes

$$\tfrac{1}{2}I\omega^2,$$

where I is the moment of inertia of the body about the axis of rotation.

EXAMPLES

1. A uniform wheel in the form of a disk arranged to rotate about the fixed polar axis through its center weighs 161 lb., and its radius is 4 ft. A light rope around the wheel exerts a force of 10 lb. tangent to the rim of the wheel for a period of 30 sec. Find the angular acceleration, the angular velocity, and the total angle described by any radius of the wheel if the wheel starts from rest. Also find its kinetic energy at the end of the time interval, and show that it is equal to the work done by the force of 10 lb.

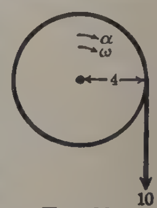

FIG. 293

Solution. Taking moments about the axis of rotation, the equation $N = I\alpha$ gives

$$40 = \frac{1}{2} \times \frac{161}{32.2} \times (4)^2 \times \alpha,$$

from which $\alpha = 1$ radian per second per second.

Also, since $\omega = \alpha t$ and $\theta = \frac{1}{2}\alpha t^2$,

$$\omega = 30 \text{ radians per second} \quad \text{and} \quad \theta = 450 \text{ radians.}$$

The angular velocity is $\frac{30}{2\pi}$ revolutions per second, and the total number of revolutions is $\frac{450}{2\pi}$. The kinetic energy is

$$\frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{1}{2} \times \frac{161}{32.2} \times (4)^2 \times (30)^2 = 18{,}000 \text{ ft.-lb.}$$

The work done by the external force of 10 lb. is the product of the force by the distance through which it moves, or

$$10 \times 450 \times 4 = 18{,}000 \text{ ft.-lb.}$$

2. A thin rod of weight W and length l rotates in a vertical plane about a fixed axis through one end. If its initial angular velocity is ω_0 in the vertical position, find its angular velocity when it has fallen through an angle θ.

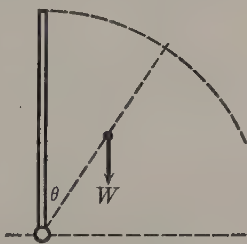

FIG. 294

Solution. First method. Taking moments about the axis of rotation O, the general equation $N = I\alpha$ gives

$$\frac{Wl}{2} \sin \theta = \frac{1}{3} \frac{Wl^2}{g} \frac{d^2\theta}{dt^2}.$$

Multiplying both sides of the equation by $\frac{d\theta}{dt}$ and integrating,

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = -\frac{3\,g \cos \theta}{2\,l} + C.$$

Also, since $\theta = 0$ when $\frac{d\theta}{dt} = \omega_0$,

$$C = \frac{1}{2}\,\omega_0^2 + \frac{3\,g}{2\,l},$$

and

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{3\,g}{2\,l}(1 - \cos \theta) + \frac{1}{2}\,\omega_0^2.$$

This equation determines the angular velocity $\frac{d\theta}{dt}$ after the rod has fallen through any angle θ.

Second method. By the method of work and energy the initial kinetic energy is

$$\frac{1}{2} I\omega_0^2 = \frac{1}{2} \times \frac{1}{3} \frac{W}{g}\, l^2\omega_0^2.$$

The work of gravity is the weight of the rod multiplied by the vertical distance through which the center of gravity descends, or $\frac{Wl}{2}(1 - \cos \theta)$. Hence

$$\frac{1}{6} \frac{W}{g}\, l^2\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{6} \frac{W}{g}\, l^2\omega_0^2 + \frac{Wl}{2}(1 - \cos \theta),$$

or

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{3\,g}{2\,l}(1 - \cos \theta) + \frac{1}{2}\,\omega_0^2,$$

as before.

3. A string of negligible weight is wrapped around a cone pulley weighing 32 lb., having a radius of 2 ft. and a radius of gyration of 2.5 ft. The end of the string is attached to a body weighing 15 lb. Find the acceleration of the 15-pound weight, friction being neglected.

Solution. Let y be the distance of the 15-pound weight below the center of the pulley, y being taken as positive downward. Then, if T is the tension in the cord (not 15 lb.), the equation of motion of the 15-pound weight is, by § 142,

$$15 - T = \frac{15}{32.2}\frac{d^2y}{dt^2}. \qquad (1)$$

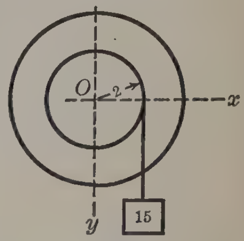

FIG. 295

The equation of rotation, $N = I\alpha$, gives

$$2\,T = \frac{32}{32.2}(2.5)^2\frac{d^2\theta}{dt^2}. \qquad (2)$$

Since the string does not slip on the surface of the pulley,

$$2\frac{d\theta}{dt} = \frac{dy}{dt},$$

and hence

$$2\frac{d^2\theta}{dt^2} = \frac{d^2y}{dt^2}. \qquad (3)$$

Equations (1) and (2) are the dynamical equations and (3) is the geometric equation necessary for the solution of the problem.

Eliminating T between (1) and (2) and replacing $\frac{d^2\theta}{dt^2}$ by its value from (3),

$$\frac{d^2y}{dt^2} = 7.43 \text{ ft. per second per second.}$$

PROBLEMS

1. Let the angle θ be measured from the horizontal axis, instead of from the vertical axis as in Example 2, p. 257. Show that the equation of motion is $\frac{Wl}{2}\cos\theta = -\frac{1}{3}\frac{Wl^2}{g}\frac{d^2\theta}{dt^2}$, and explain why the second member is preceded by a negative sign.

2. A turbine runner, weighing 40 T. and having a radius of gyration of 3 ft., is running idle at 3000 R.P.M. If air resistance and bearing friction together are assumed to be equal to a constant force of 200 lb. at a radius of 5 ft., find the time required for the runner to come to rest. *Ans.* 1.95 hr.

3. A trapdoor 4 ft. square is hinged at one edge. If it is lifted into a vertical position and slightly displaced, find the velocity with which the outer edge strikes the floor, air resistance and friction being neglected. (Use the formula $N = I\alpha$ and check the results by the energy method.) *Ans.* 19.64 ft. per second.

4. The wheel of a wheel and axle has a diameter of 8 ft., and the axle has a diameter of 4 ft. A light rope wound around the wheel supports a weight of 100 lb., and a second rope, wound in the opposite direction around the axle, supports a weight of 160 lb. Bearing friction is equivalent to a constant moment of 40 lb.-ft. The wheel and axle weighs 200 lb., and its radius of gyration is 3 ft. Find the accelerations of the weights.

Ans. 1.27 ft. per second per second,
0.63 ft. per second per second.

5. Two concentric disks, one 4 ft. in diameter, weighing 20 lb., and the other 2 ft. in diameter, weighing 5 lb., are bolted together and mounted on a horizontal axis having a constant frictional moment of 5 lb.-ft. A cord attached to the movable end of a fixed spring whose modulus is 10 lb. per inch is wound on the smaller disk until the spring is stretched 12 in. and then released. Find the angular velocity imparted to the disks when the spring is just unstretched. If the cord is detached at that instant, find the time for the frictional moment to bring the disks to rest, and find the number of revolutions during that interval.

Ans. $\omega = 9.14$ radians per second,
$\alpha = 3.79$ radians per second per second,
$t = 2.4$ sec., 1.75 revolutions.

6. A string which passes over a rough pulley of weight W_1 carries two masses, of weights W_2 and W_3, at its ends. If $W_2 > W_3$, and if r and k are the radius and the radius of gyration of the pulley, find the acceleration of the weights, assuming that friction is neglected (Atwood's machine). (Solve also by the energy equation, assuming that the weight W_2 descends through a distance s. Make use of the relation $v^2 = 2\,as$ to eliminate s. Notice that this relation is valid only when the acceleration is constant.)

$$\textit{Ans. } a = \frac{g(W_2 - W_3)}{W_2 + W_3 + \dfrac{W_1 k^2}{r^2}}.$$

7. A horizontal revolving table is driven at a constant speed of 60 R.P.M. A wheel of radius 2 ft., whose weight of 12 lb. is concentrated in its rim, is placed on a vertical shaft at the center of the table so that only the rim contacts the table. Find the time for the wheel to acquire the angular velocity of the table. ($\mu = 0.1$.)

Ans. 3.9 sec.

8. A disk whose weight is 644 lb. and whose diameter is 12 ft. is mounted on a frictionless horizontal axle. A concentrated weight of 322 lb. is attached to the disk at a point 4 ft. above the axle. If the system is displaced from its position of unstable equilibrium, find its maximum angular velocity. *Ans.* 3.15 radians per second.

9. A weight of 10 lb. resting on a rough horizontal plane ($\mu = 0.2$) is connected to a weight of 12 lb. by a light cord passing over a rough pulley of 1-foot radius. The weight of the pulley is 8 lb., and its radius of gyration is 0.75 ft. Assuming that the moment of friction at the axle of the pulley is 2 lb.-ft., find the acceleration of the weights and the tensions in the cords.

Ans. $a = 9.72$, $T_1 = 8.38$, $T_2 = 5.02$.

10. Forty feet of rope weighing 1 lb. per foot hangs at rest in two equal parts over a rough pulley. The diameter of the pulley is 10 ft., its weight is 64.4 lb., its radius of gyration is 4 ft., and the moment of its axle friction is constant and equal to 25 lb.-ft. A weight of 25 lb. is attached to one end of the rope. Find the angular velocity of the pulley when the weight has descended a distance of 10 ft. *Ans.* 2.70 radians per second.

11. A heavy wheel attached to a fan is observed to rotate at 180 R.P.M. at a certain instant and at 60 R.P.M. 2 min. later. The flywheel and fan weigh 500 lb., and the radius of gyration of their combined mass is 2 ft. Find the retarding couple when the wheel is rotating at 120 R.P.M. if bearing friction is neglected and if the resistance of the air varies as the square of the angular velocity.

HINT. Let the moment of the air resistance be $k\omega^2$; then $N = I\alpha$ gives $k\omega^2 = -I\dfrac{d\omega}{dt}$, which, when integrated, becomes $kt + c = \dfrac{I}{\omega}$.

Ans. 8.67 lb.-ft.

12. A flywheel weighing 200 lb. and having a radius of gyration of 2 ft. is being slowed down by a retarding couple which varies directly as the angular velocity. The angular velocity is observed to drop from 120 R.P.M. to 60 R.P.M. in 1 min. Find the retarding couple at the beginning and at the end of the time interval.

Ans. 3.62 lb.-ft., 1.81 lb.-ft.

164. The compound pendulum. A rigid body which swings about a fixed horizontal axis under the action of gravity is called a *compound pendulum*. The time of oscillation of a compound pendulum may be found as follows:

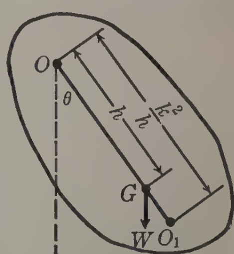

FIG. 296

Let a vertical plane containing the axis O be selected as a reference plane, and let a second plane containing the axis O and the center of gravity G make an angle θ with the reference plane. The equation of

motion, $N = I\frac{d^2\theta}{dt^2}$ (§ 162), becomes

$$Wh \sin \theta = -\frac{W}{g} k^2 \frac{d^2\theta}{dt^2}, \tag{1}$$

where W is the weight of the pendulum, h is the distance from the fixed axis O to the center of gravity G, and k is the radius of gyration with respect to the axis. If the angle θ is small, the sine of θ may be replaced by the angle θ, and (1) becomes

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{k^2}\theta. \tag{2}$$

This is the equation of simple harmonic motion (§ 146), and the period of a complete oscillation is therefore

$$T = 2\pi\sqrt{\frac{k^2}{gh}}. \tag{3}$$

The period of a simple pendulum is (§ 152)

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Therefore a compound pendulum has exactly the same period of oscillation as a simple pendulum of length $\frac{k^2}{h}$. The length $\frac{k^2}{h}$ is called the length of the simple equivalent pendulum.

165. The center of oscillation of a compound pendulum. If a length $OO_1 = \frac{k^2}{h}$ is measured from the center of suspension O along the line OG, Fig. 297, the point O_1 is called the center of oscillation, because the body has the same period of oscillation about the axis O_1 as it has about the axis O. This property of the center of oscillation may be shown as follows:

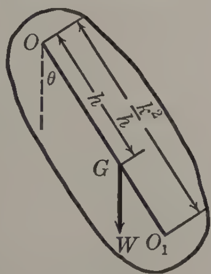

FIG. 297

Let k_1 be the radius of gyration of the pendulum about the center of oscillation O_1. Then, by the parallel-axis theorem (§ 112),

$$k_1^2 = k^2 - \overline{OG}^2 + \overline{O_1G}^2$$
$$= k^2 - (OG + O_1G)(OG - O_1G). \tag{1}$$

Also $$\frac{k^2}{OG} = OO_1 = (OG + O_1G). \tag{2}$$

Hence $$k_1^2 = k^2\left[1 - \frac{1}{OG}(OG - O_1G)\right] = \frac{k^2 O_1G}{OG}. \tag{3}$$

But by (3), § 164, the period of oscillation of the pendulum about O_1 is

$$T_1 = 2\,\pi\sqrt{\frac{k_1{}^2}{g \cdot O_1G}}. \tag{4}$$

Substituting the value of $k_1{}^2$ from (3) in (4),

$$T_1 = 2\,\pi\sqrt{\frac{k^2}{g \cdot OG}} = 2\,\pi\sqrt{\frac{k^2}{gh}},$$

which is the same as the period of oscillation about the axis O_1 (§ 164).

166. The torsion pendulum. Let a long wire be suspended from a fixed support A, and let a heavy disk B be attached concentric with the wire at its lower end C. If the body B is rotated through an angle θ and released, the elastic forces set up in the wire by the twist will cause the disk to oscillate. If the angle of twist is not too great, the elastic restoring couple will be proportional to the angle of twist. The couple may therefore be expressed by $C\theta$, where C is a constant. If friction is neglected, the equation of motion is

FIG. 298

$$C\theta = -\frac{W}{g}k^2\frac{d^2\theta}{dt^2}, \tag{1}$$

where W is the weight of the disk B and k is its radius of gyration about the axis of suspension. Equation (1) may be written

$$\frac{d^2\theta}{dt^2} = -\frac{Cg}{Wk^2}\,\theta. \tag{2}$$

Equation (2) is the equation of simple harmonic motion (§ 146), and the time of oscillation is therefore

$$T = 2\,\pi\sqrt{\frac{Wk^2}{Cg}}. \tag{3}$$

Let N be the torque required to twist the wire through an arbitrary angle θ, measured in radians. Then $C = \frac{N}{\theta}$, and (3) becomes

$$T = 2\,\pi\sqrt{\frac{Wk^2\theta}{gN}}. \tag{4}$$

The moment, or torque, N required to give the wire an angle of twist θ may be measured experimentally.

167. Moment of inertia determined by the torsion pendulum. The moment of inertia I_1 of the disk of a torsion pendulum may be calculated from its weight and dimensions or may be otherwise known. The period of the pendulum is

$$T_1 = 2\,\pi\sqrt{\frac{I_1\theta}{N}}. \tag{1}$$

Let the body whose moment of inertia I_2 about a particular gravity axis is required be attached to the disk so that the gravity axis and wire coincide. Let the new period be T_2, where

$$T_2 = 2\,\pi\sqrt{\frac{(I_1 + I_2)\theta}{N}}. \tag{2}$$

Dividing (2) by (1) and solving for I_2 gives

$$I_2 = \left(\frac{T_2{}^2 - T_1{}^2}{T_1{}^2}\right) I_1.$$

The moment of inertia I_2 is thus expressed in terms of computed and observed quantities.

If the value of I_1 is not readily available by direct measurement, it may be obtained from equation (4), § 166, by measuring the angle θ corresponding to a measured torque N and by observing the period of oscillation T.

PROBLEMS

1. A cast-iron cube whose edge is 1 ft. long oscillates about a horizontal axis coinciding with one of its edges. Find the length of the simple equivalent pendulum. Does the result depend upon the material of the cube?
Ans. $\frac{2\sqrt{2}}{3}$ ft.

2. A circular disk of radius r oscillates about a horizontal axis perpendicular to the face of the disk and passing through a point on its circumference. Find the length of the simple equivalent pendulum and the period of oscillation.
Ans. $l = 1.5\,r$, $t = 2\,\pi\sqrt{\frac{1.5\,r}{g}}$.

3. A hemisphere oscillates about an axis passing through its vertex parallel to the base. Find the length of the simple equivalent pendulum.
Ans. $\frac{26}{25}\,r$.

4. A right circular cone of radius r and height h oscillates about a horizontal axis parallel to a diameter of the base through the vertex. Find the length of the simple equivalent pendulum.
Ans. $\frac{r^2 + 4\,h^2}{5\,h}$.

5. A cast-iron cube whose edge is 1 ft. long is suspended as a torsion pendulum by a wire attached to a vertex. The period of oscillation is observed to be 2 sec. Find the couple necessary to twist the wire through 180°. *Ans.* 72.15 lb.-ft.

6. A moment of 5 lb.-ft. is required to twist the disk of a torsion pendulum through an angle of 15°. Find the period of oscillation if the moment of inertia of the disk is I. *Ans.* $t = \pi \sqrt{\frac{\pi I}{15}}$.

7. A flywheel is fastened to the free end of a shaft, the other end being fixed. A moment of 60 lb.-ft. is required to twist the shaft through 0.05 radian, and the time of oscillation is 4 sec. Find the moment of inertia of the flywheel. *Ans.* 486.3 slug-ft.-ft.

8. The disk of a torsion pendulum has a moment of inertia of 4 slug-ft.-ft., and its period of oscillation is 0.5 sec. What is the moment of inertia of a second disk placed on the first if the period of vibration for both disks is 1.5 sec.? *Ans.* $I = 32$ slug-ft.-ft.

168. Forces at the axis of rotation of a compound pendulum. The simple case in which the body and the forces are symmetrical with respect to the plane perpendicular to the axis containing the center of gravity will be considered first.

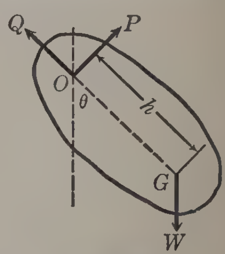

FIG. 299

The center of gravity G describes a circle about the axis at O, and therefore the acceleration of that point may be represented by two components, one along the radius and the other along the tangent given, respectively, by $h\left(\frac{d\theta}{dt}\right)^2$ and $h\frac{d^2\theta}{dt^2}$. Moreover, since the motion of the center of gravity is the same as if all the external forces and the mass were collected at that point (§ 158), the equations of motion are

$$Q - W\cos\theta = \frac{W}{g}h\left(\frac{d\theta}{dt}\right)^2, \tag{1}$$

$$P - W\sin\theta = \frac{W}{g}h\frac{d^2\theta}{dt^2}, \tag{2}$$

where P and Q are the components of the hinge reaction at O. Equations (1) and (2) are obtained by setting the components of the external forces along and perpendicular to the line OG equal to the mass of the body multiplied by the components of the acceleration in those directions.

The moment equation $N = I\alpha$ gives

$$W \sin\theta\, h = -\frac{W}{g} k^2 \frac{d^2\theta}{dt^2}, \tag{3}$$

where k is the radius of gyration about the axis of rotation.

The component P of the hinge reaction at O is obtained by eliminating $\frac{d^2\theta}{dt^2}$ between (2) and (3). Thus

$$P = W \sin\theta \left(1 - \frac{h^2}{k^2}\right). \tag{4}$$

The value of P is therefore independent of the initial conditions of the motion; it depends only upon the value of the sine of the angle, and it becomes zero when the center of gravity is in the vertical through O.

The angular velocity $\frac{d\theta}{dt}$ may be found by integrating (3). The value of Q is found from (1) after $\frac{d\theta}{dt}$ has been determined from (3). It is usually easier to obtain the value of $\frac{d\theta}{dt}$ from the energy equation, as in the solution of the example on page 266.

EXAMPLE

A circular disk weighing 21 lb., and having a radius of 2 ft., is mounted on a horizontal axis perpendicular to its face through a point on its circumference. The initial conditions are $\omega = 0$ when $\theta = 180°$, θ being the angle the disk has turned through from its lowest position. Find the components of the hinge reaction in terms of θ, and evaluate them when $\theta = 0$ and when $\theta = 90°$.

FIG. 300

Solution. The moment of inertia about O is $\frac{3}{2}\frac{W}{g}r^2$; also $h = r$. Hence (1), (2), and (3), § 168, may be written.

$$Q - W\cos\theta = \frac{W}{g} r \left(\frac{d\theta}{dt}\right)^2, \tag{1}$$

$$P - W\sin\theta = \frac{W}{g} r \frac{d^2\theta}{dt^2}, \tag{2}$$

and

$$\sin\theta = -\frac{3}{2}\frac{r}{g}\frac{d^2\theta}{dt^2}. \tag{3}$$

From (3),

$$\frac{d^2\theta}{dt^2} = -\frac{2}{3}\frac{g}{r}\sin\theta.$$

Multiplying both sides by $\frac{d\theta}{dt}$ and integrating,

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{2}{3}\frac{g}{r}\cos\theta + C.$$

From the initial conditions, $\frac{d\theta}{dt} = 0$ when $\theta = \pi$. Hence

$$C = \frac{2}{3}\frac{g}{r}$$

and
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{4}{3}\frac{g}{r}(1 + \cos\theta). \tag{4}$$

The same result may be obtained from the energy equation as follows: The work of gravity when θ changes from π to θ is

$$Wr(1 + \cos\theta) = \frac{1}{2}I\omega^2 = \frac{1}{2}\cdot\frac{3}{2}\frac{W}{g}r^2\left(\frac{d\theta}{dt}\right)^2,$$

and hence
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{4}{3}\frac{g}{r}(1 + \cos\theta),$$

as before.

Substituting the value of $\frac{d\theta}{dt}$ in (1),

$$Q = \frac{W}{3}(4 + 7\cos\theta). \tag{5}$$

Eliminating $\frac{d^2\theta}{dt^2}$ between (2) and (3),

$$P = \frac{W}{3}\sin\theta. \tag{6}$$

The total bearing reaction $R = \sqrt{P^2 + Q^2}$. The values of P, Q, and R when $\theta = 0°$ are

$$0, \quad \frac{11}{3}W, \quad \text{and} \quad \frac{11}{3}W.$$

The corresponding values when $\theta = 90°$ are

$$\frac{W}{3}, \quad \frac{4\,W}{3}, \quad \text{and} \quad \frac{\sqrt{17}}{3}W.$$

PROBLEMS

1. Solve the example on page 265, assuming θ to be measured from the upward vertical through O.
Ans. $\frac{W}{3}(4 - 7\cos\theta)$, $\frac{W}{3}\sin\theta$.

2. A slender rod, of weight W and length $2\,a$, is hinged at one end. Find the forces P and Q (§ 168) in any position and also when $\theta = 90°$ and when $\theta = 0°$. Assume that the rod starts from rest with the center of gravity vertically above the axis of rotation.

Ans. $P = \frac{W}{4}\sin\theta$, $Q = \frac{W}{2}(3 + 5\cos\theta)$;
when $\theta = 0$, $P = 0$ and $Q = 4\,W$;
when $\theta = 90°$, $P = \frac{W}{4}$ and $Q = \frac{3}{2}W$.

3. A square plate of side $2\,a$ ft., weighing W lb., is fixed to a vertical axis, with one edge coincident with the axis, by two pins placed

at the corners. If the plate rotates with a uniform angular velocity of n revolutions per minute, show that the tension in the upper pin is

$$\frac{W}{2}\left(1+\frac{an^2\pi^2}{28{,}980}\right).$$

4. A semicircular plate of weight W and radius r is attached to a vertical axis by two pins at the extremities of a diameter. Find the horizontal force in the lower pin if the angular velocity ω is constant.

Ans. $\frac{2}{3}\frac{W}{\pi}\left(1-\frac{\omega^2}{g}\right)$.

5. A thin rod 6 ft. long, weighing 21 lb., is attached to the ceiling of a motor truck by a hinge. If the brakes are applied so that a negative acceleration of $\frac{g}{3}$ is produced, find the angular velocity of the rod when it makes an angle of 30° with the vertical. Also find the vertical and horizontal components of the hinge reaction.

Ans. $\omega = 0.725$ radian per second, 20.22 lb., 10.40 lb.

169. Reactions on the axis of rotation; pressure on bearings. Let any body be rigidly attached to an axis, the axis being free to rotate in two fixed bearings. The body being subjected to

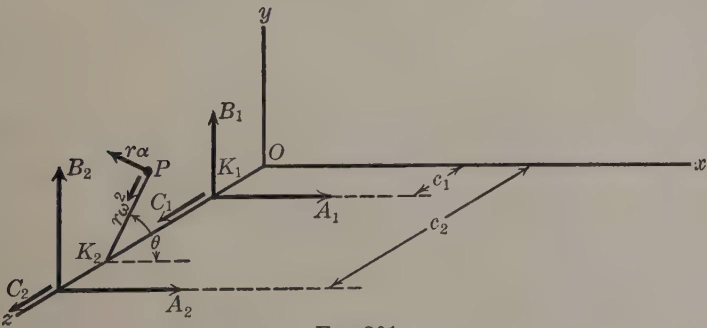

FIG. 301

the action of any given forces, it is required to find the reactions on the bearings.

Let the z axis be taken as the axis of rotation, and let the bearings be placed at $K_1(0, 0, c_1)$ and $K_2(0, 0, c_2)$. Also let the components of the bearing reactions at K_1 and K_2 be A_1, B_1, C_1 and A_2, B_2, C_2 respectively. Let any particle of mass m be located at $P(x, y, z)$, and let (r, θ) be the polar coördinates of P in the plane through P perpendicular to the z axis. Since the particle P describes a circle of radius r, the external forces necessary to produce the rotation are $m\omega^2 r$ toward the center of the

circle and $mr\alpha$ tangent to the circle. The components of these forces along the x and y axes are, respectively:

$$-m\omega^2 r\cos\theta - mr\alpha\sin\theta \quad \text{and} \quad -m\omega^2 r\sin\theta + mr\alpha\cos\theta,$$

or

$$-m\omega^2 x - m\alpha y \quad \text{and} \quad -m\omega^2 y + m\alpha x.$$

Hence the equations of motion of the body are

$$\Sigma X + A_1 + A_2 = \Sigma(-mx\omega^2 - my\alpha), \tag{1}$$

$$\Sigma Y + B_1 + B_2 = \Sigma(-my\omega^2 + mx\alpha), \tag{2}$$

$$\Sigma Z + C_1 + C_2 = 0, \tag{3}$$

$$\Sigma(Zy - Yz) - B_1c_1 - B_2c_2 = \Sigma(m\omega^2 yz - m\alpha xz), \tag{4}$$

$$\Sigma(Xz - Zx) + A_1c_1 + A_2c_2 = \Sigma(-m\omega^2 xz - m\alpha yz), \tag{5}$$

$$\Sigma(Yx - Xy) = \Sigma mr^2\alpha. \tag{6}$$

In these equations ΣX, ΣY, and ΣZ are *given* external forces, and $\Sigma(Zy - Yz)$, $\Sigma(Xz - Zx)$, and $\Sigma(Yx - Xy)$ are *given* external couples.

The weight of the body would enter these equations as an external force and give rise to a portion of the bearing pressure; but since these are easily determined by statics, they are hereafter omitted. Friction at the bearings may also be omitted, since it merely diminishes the given couple $\Sigma(Yx - Xy)$.

Since ω and α, at any instant, are the same for every particle,

$$\Sigma(mx\omega^2) = \omega^2\bar{x}M, \quad \Sigma(my\alpha) = \alpha\bar{y}M, \text{ etc.}$$

Making these substitutions, (1) to (6) become

$$\Sigma X + A_1 + A_2 = -M\bar{x}\omega^2 - M\bar{y}\alpha, \tag{7}$$

$$\Sigma Y + B_1 + B_2 = -M\bar{y}\omega^2 + M\bar{x}\alpha, \tag{8}$$

$$\Sigma Z + C_1 + C_2 = 0, \tag{9}$$

$$\Sigma(Zy - Yz) - B_1c_1 - B_2c_2 = \omega^2\Sigma(myz) - \alpha\Sigma(mxz), \tag{10}$$

$$\Sigma(Xz - Zx) + A_1c_1 + A_2c_2 = -\omega^2\Sigma(mxz) - \alpha\Sigma(myz), \tag{11}$$

$$\Sigma(Yx - Xy) = \alpha\Sigma(mr^2) = \alpha I. \tag{12}$$

Since the body is given, the mass M, the products of inertia $\Sigma(myz)$ and $\Sigma(mxz)$, the moment of inertia $\Sigma(mr^2)$, and $\bar{x}$ and $\bar{y}$ can be obtained.

The equations can be solved as follows: The angular acceleration α is given from (12), and by integration ω is obtained. Substitution of these values in (7), (8), (10), and (11) gives four equations for determining the bearing pressures A_1, B_1, A_2, and

B_2. The thrusts, C_1 and C_2, depend on the adjustment and type of the bearings. Equation (9) gives the sum of the thrusts.

170. Static and dynamic balance. When any body or system of bodies, including the axis, is mounted in two frictionless bearings and is at rest in a position of equilibrium under the action of gravity only, the pressure on each bearing is easily computed by the principles of statics. When the body is rotated to any other position and freed and remains in that position, it is said to be in static or standing balance. If the body is rotating with or without the action of a couple and there is no pressure on the bearings other than the pressure due to its weight, the body is said to be in dynamic or running balance.

Evidently the body under the action of only *gravity* and the *rotating couple* will be in both static and dynamic balance if the pressures on the bearings other than those due to gravity are made zero.

Introducing these conditions in equations (7), (8), (10), and (11) of § 169 gives

$$\omega^2\bar{x} + \alpha\bar{y} = 0, \tag{1}$$

$$\omega^2\bar{y} - \alpha\bar{x} = 0, \tag{2}$$

$$\omega^2\Sigma(myz) - \alpha\Sigma(mxz) = 0, \tag{3}$$

$$\omega^2\Sigma(mxz) + \alpha\Sigma(myz) = 0. \tag{4}$$

Equation (12) becomes

$$I\alpha = N, \tag{5}$$

where N is the given couple. If $N = 0$, there is no angular acceleration.

Since the body is to be in balance for any speed or acceleration, (1) to (4) must be satisfied for all values of ω^2 and α. Hence

$$\bar{x} = 0, \tag{6}$$

$$\bar{y} = 0, \tag{7}$$

$$\Sigma(mxz) = 0, \tag{8}$$

$$\Sigma(myz) = 0, \tag{9}$$

which are the analytic specifications for static and dynamic balance.

Equations (6) and (7) require the center of gravity of the body to lie on the axis of rotation.

Equations (8) and (9) require the product of inertia of the body with respect to the xy and yz planes and with respect to the xy and xz planes each to be zero.

Given any number of masses located at definite points; to find whether a single mass m_1 at (x_1, y_1, z_1) will produce static and dynamic balance.

Let the sums of the moments of the given masses be represented by $\Sigma(mx)$ and $\Sigma(my)$, and the sums of the products of inertia by $\Sigma(mxz)$ and $\Sigma(myz)$.

The equations for finding m_1 at (x_1, y_1, z_1) for balance are

$$\Sigma(mx) + m_1x_1 = 0, \tag{10}$$

$$\Sigma(my) + m_1y_1 = 0, \tag{11}$$

$$\Sigma(mxz) + m_1x_1z_1 = 0, \tag{12}$$

$$\Sigma(myz) + m_1y_1z_1 = 0. \tag{13}$$

Eliminating m_1 from the first two equations,

$$\frac{x_1}{y_1} = \frac{\Sigma(mx)}{\Sigma(my)}. \tag{14}$$

Eliminating m_1 from the last two equations,

$$\frac{x_1}{y_1} = \frac{\Sigma(mxz)}{\Sigma(myz)}. \tag{15}$$

Equations (14) and (15) are in general inconsistent. Equations (10), (11), (12), and (13) cannot be satisfied unless

$$\frac{\Sigma(mx)}{\Sigma(my)} = \frac{\Sigma(mxz)}{\Sigma(myz)},$$

which is a relation between arbitrary given quantities and hence not true in general. Hence the system cannot in general be balanced by one mass.

Similar equations are obtained by introducing a second mass m_2 at (x_2, y_2, z_2). They are

$$\Sigma(mx) + m_1x_1 + m_2x_2 = 0,$$

$$\Sigma(my) + m_1y_1 + m_2y_2 = 0,$$

$$\Sigma(mxz) + m_1x_1z_1 + m_2x_2z_2 = 0,$$

$$\Sigma(myz) + m_1y_1z_1 + m_2y_2z_2 = 0.$$

These four equations in eight unknowns can be satisfied in several ways. For example, assume four of the unknowns, x_1, y_1, x_2, y_2, arbitrarily; then the first two equations are linear in m_1 and m_2. Substituting the values in the last two, they are linear in z_1 and z_2. The values of z might be such as to place the weights beyond the end of the shaft or in undesirable lo-

cations on the shaft. Therefore z_1 and z_2 are usually assumed. The four equations determine the four unknowns, namely (m_1x_1), (m_2x_2), (m_1y_1), and (m_2y_2). Two more unknowns are then assumed, for example, x_1 and x_2 or m_1 and m_2, but not x_1 and y_1. Hence it follows that any body or system of bodies may be balanced both statically and dynamically by the addition or removal of two properly located masses.

EXAMPLE

Let it be required to balance the rotating weights of Fig. 302 by weights placed in the planes A and B at radial distances of 0.5 ft. and 0.25 ft. respectively.

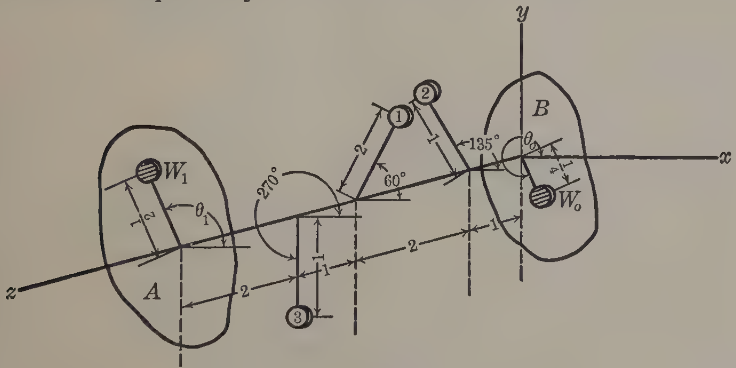

FIG. 302

Solution. Let the axis of rotation be chosen as the z axis, and let the x and y axes be fixed in the plane of rotation B of one of the balancing weights. Also let θ_0 and θ_1 be the angles which the balancing weights W_0 and W_1 make with the x direction. In order that the system may be in balance, it is necessary (§ 170) that

$$\Sigma(mxz) = 0, \quad \Sigma(myz) = 0,$$
$$\Sigma(mx) = 0, \quad \Sigma(my) = 0.$$

Since these equations are homogeneous in m, the weight W may be used instead of m. The computation is facilitated by a tabular arrangement.

LOAD	x	y	z	Wxz	Wyz	Wx	Wy
2	$-\frac{\sqrt{2}}{2}$	$+\frac{\sqrt{2}}{2}$	$+1$	$-\sqrt{2}$	$+\sqrt{2}$	$-\sqrt{2}$	$+\sqrt{2}$
1	$+1$	$+\sqrt{3}$	$+3$	$+3$	$+3\sqrt{3}$	$+1$	$+\sqrt{3}$
3	0	-1	$+4$	0	-12	0	-3
W_0	$\frac{1}{4}\cos\theta_0$	$\frac{1}{4}\sin\theta_0$	0	0	0	$\frac{1}{4}W_0\cos\theta_0$	$\frac{1}{4}W_0\sin\theta_0$
W_1	$\frac{1}{2}\cos\theta_1$	$\frac{1}{2}\sin\theta_1$	6	$3W_1\cos\theta_1$	$3W_1\sin\theta_1$	$\frac{1}{2}W_1\cos\theta_1$	$\frac{1}{2}W_1\sin\theta_1$

The condition $(\Sigma Wxz) = 0$ requires that

$$3\ W_1 \cos\theta_1 + 3 - \sqrt{2} = 0, \tag{1}$$

and the condition $\Sigma(Wyz) = 0$ requires that

$$3\ W_1 \sin\theta_1 - 12 + 3\sqrt{3} + \sqrt{2} = 0. \tag{2}$$

Also $\Sigma(Wx) = 0$ and $\Sigma(Wy) = 0$ lead to the equations

$$\tfrac{1}{2}\ W_1 \cos\theta_1 + \tfrac{1}{4}\ W_0 \cos\theta_0 + 1 - \sqrt{2} = 0, \tag{3}$$

$$\tfrac{1}{2}\ W_1 \sin\theta_1 + \tfrac{1}{4}\ W_0 \sin\theta_0 - 3 + \sqrt{3} + \sqrt{2} = 0. \tag{4}$$

Solving (1) and (2) for W_1 and θ_1,

$$W_1 = 1.87 \text{ lb.}$$

and

$$\theta_1 = 106^\circ\ 24'.$$

Substituting the values of θ_1 and W_1 in (3) and (4) and solving,

$$W_0 = 4.98 \text{ lb.}$$

and

$$\theta_0 = 303^\circ\ 0'.$$

PROBLEM

Find the positions (x and y coördinates) of two weights, of 4 lb. and 5 lb., in planes A and B, respectively, to balance the system of rotating weights shown in Fig. 303. *Ans.* $x_4 = -0.768$ ft., $y_4 = -0.325$ ft.; $x_5 = 0.236$ ft., $y_5 = -0.381$ ft.

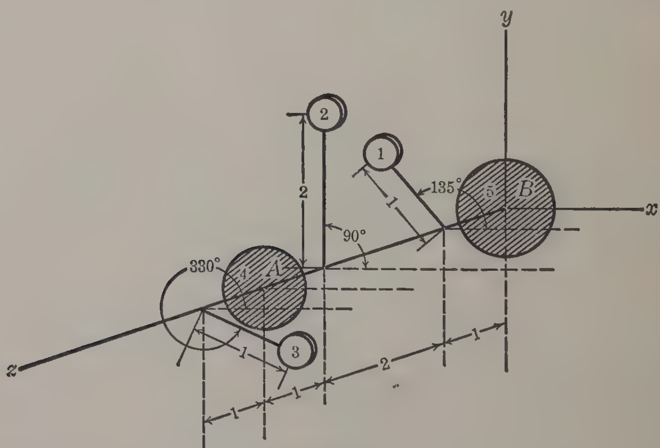

FIG. 303

171. The governor. The governor is a mechanical device for regulating the speed of prime movers. It ordinarily consists of a system of weights which are caused to rotate about a fixed axis at a definite fraction of the speed of the machine to be

governed. A change in speed of the governor causes the weights to move away from or approach the axis of rotation owing to the change in centrifugal force. The motion of the weights operates mechanical means for controlling the amount or pressure of the working fluid. In the case of a steam engine, the governor may open and close a throttle valve or it may alter the cut-off of the valve gear.

172. Simple Watt governor. The simple Watt governor consists of two balls A which are caused to rotate in a horizontal plane about the vertical axis BC. If friction and the weights of the arms are neglected, the governor becomes a conical pendulum, and hence from (3), § 150, the height h at which the governor operates at angular velocity ω is

$$h = \frac{g}{\omega^2}, \tag{1}$$

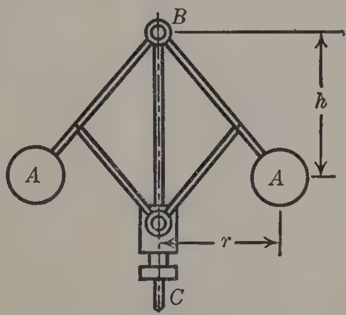

FIG. 304

where $\omega = \frac{2\pi n}{60}$ and n is the number of revolutions per minute. It follows from (1) that the height h of a Watt governor is inversely proportional to the square of the angular velocity and is independent of the size and weight of the balls or the length of the arms.

Differentiating (1),

$$dh = -\frac{2g}{\omega^3}\,d\omega, \tag{2}$$

from which it is evident that a given change in speed $d\omega$ is much more effective in changing the height h at low speeds than at high speeds. This type of governor is therefore not adapted to high speeds, and it has the further disadvantage that the work which it is capable of performing in moving the governor gear is comparatively small. The work of the governor is the product of the weight of the balls $2W$ and the change in h.

173. The Porter governor. The Porter governor is a Watt governor to which an additional weight W_1 is attached, as shown in Fig. 305. Let W be the weight of each ball. By the theory of virtual work, § 103,

$$2W\,dh + W_1\,2\,dh + 2\frac{W}{g}\omega^2 r\,dr = 0. \tag{1}$$

Also, from the figure,

$$r^2 + h^2 = l^2$$

and
$$r\,dr = -h\,dh. \tag{2}$$

Substituting (2) in (1),

$$W + W_1 = \frac{W}{g}\,\omega^2 h \tag{3}$$

and
$$h = \frac{W + W_1}{W}\,\frac{g}{\omega^2}. \tag{4}$$

If the friction F is included as a force acting with the weight W when the balls are on the point of moving outward, and in the opposite direction when the balls are about to move inward, an analysis by virtual work gives

$$h_1 = \frac{W + W_1 + F}{W}\,\frac{g}{\omega_1{}^2} \tag{5}$$

and
$$h_2 = \frac{W + W_1 - F}{W}\,\frac{g}{\omega_2{}^2}. \tag{6}$$

If h_1 and h_2 are the same,

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{W + W_1 + F}{W + W_1 - F}}. \tag{7}$$

The fraction $\frac{\omega_1}{\omega_2}$ is the ratio between the extreme speeds at which the governor is prevented from operating by friction.

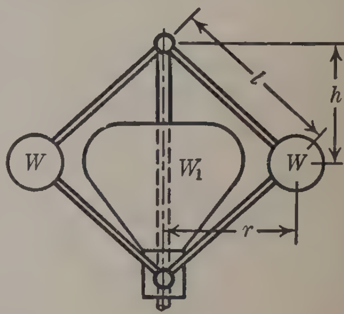

FIG. 305

174. Isochronous governors. Let f be the horizontal controlling force on the balls of a governor. Also let df be the change in f which corresponds to a change dr in the radius of the ball circle r. Then, if

$$\frac{df}{f} = \frac{dr}{r},$$

the centrifugal force will change at the same rate as the controlling force, and a small variation in the speed will cause the governor to travel from one extreme position to the other. Such a governor is called isochronous. If the change in speed which is required to operate a governor over a definite range is small, the governor is said to be sensitive. If the governor assumes a definite position for a given speed, the governor is said to be stable. A truly isochronous governor is not useful in practice since it is too sensitive. The best governors are both stable and sensitive and nearly isochronous. If the governor

is too sensitive, it may give rise to continuous and rapid fluctuations in engine speed called hunting. A governor is stable if

$$\frac{df}{f} > \frac{dr}{r},$$

since the controlling force changes more rapidly than the centrifugal force.

PROBLEMS

1. The balls of a Porter governor weigh 5 lb. each, and the controlling weight weighs 120 lb. The friction is equivalent to a force of 10 lb. at the sleeve. Find the height of the balls when the speed is increased to 240 R.P.M. At what speed will the controlling weight be on the point of descending?

Ans. 1.38 ft., 222 R.P.M.

2. The governor illustrated in Fig. 306 is loaded by means of a compressed spring instead of a weight. Find the speed at which the governor must operate in order just to lift the spring if its compression is 168 lb. The balls weigh 5 lb. each, and the distances a, b, c are 6 in., 4 in., and 6 in. respectively. Neglect friction and the weight of the arms. *Ans.* 256 R.P.M.

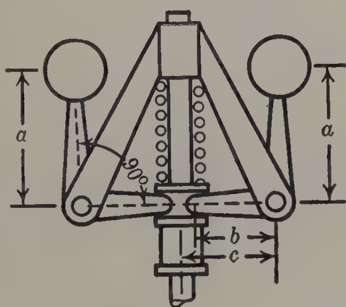

FIG. 306

3. Find the modulus of the spring required for the governor of Problem 2 if the spring is compressed $\frac{1}{10}$ in. when the speed rises to 300 R.P.M. *Ans.* 619 lb. per inch, r constant; 676 lb. per inch, r variable.

4. Show that the height in feet of a Porter governor with equal arms and links is $h = \frac{W + W_1}{W} \frac{2936}{n^2}$, where n is the speed in revolutions per minute. Neglect friction.

175. Dynamometers. Dynamometers are instruments for the measurement of power. They are classified as absorption dynamometers and transmission dynamometers. The absorption dynamometer dissipates the energy in the form of heat, while the transmission dynamometer transmits the energy without appreciable loss. The most common type of absorption dynamometer is the *Prony brake.*

The wheel C of the prime mover whose power is to be determined is clamped between two adjustable friction blocks B, attached to an arm A whose outer end rests on a scale platform. Let W_0 be the pressure on the scale platform when the adjusting nuts D are loose, and let W be the pressure when the blocks B are adjusted to obtain the desired friction. The moment of friction is $(W - W_0)l$, and the angle turned through by the wheel C in one minute is $2\pi n$ radians, where n is the number of revolutions per minute. Hence the total work absorbed is $2\pi nl(W - W_0)$ foot-pounds (§ 97), and the brake horse power is $\dfrac{2\pi nl(W - W_0)}{33{,}000}$.

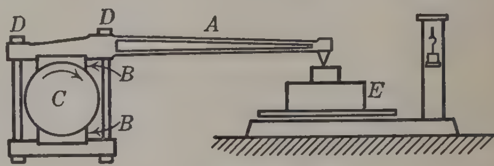

FIG. 307

One type of transmission dynamometer is the torsion dynamometer, which consists of a device for determining the twist in a measured length of a shaft that is transmitting power. Assuming Hooke's law, the twist in the shaft is proportional to the torque, or moment, transmitted by the shaft. Let N be the torque required to twist a measured length of the shaft through an angle θ, and let α be the observed angle of twist of the same length of shaft when it is under load. The torque transmitted is $\dfrac{\alpha N}{\theta}$, and the horse power transmitted is $\dfrac{\alpha N}{\theta}\dfrac{2\pi n}{33{,}000}$, where n is the number of revolutions per minute.

176. The efficiency of machines. The efficiency of a machine is defined as the ratio of the work derived from the machine to the work delivered to the machine in a given time, under the assumption that no energy is stored in the machine. On account of the work lost in friction the efficiency of a machine is always less than unity. The usefulness of certain machines requires that the friction in them be sufficient to make them irreversible under load, as, for example, the jackscrew, Weston differential hoist, and irreversible steering gear. The irreversibility of such machines depends on their efficiency. A machine is reversible or irreversible according as its efficiency is greater or less than one half. The proof of this statement is as follows:

Let F be the driving force, and let R be the external force

overcome. Also let dx and dy be the corresponding displacements of the forces F and R, and let Q be the internal work lost in friction during these displacements. Evidently

$$F\,dx = R\,dy + Q;$$

and if e is the efficiency,

$$e = \frac{R\,dy}{F\,dx} = 1 - \frac{Q}{F\,dx}.$$

When the driving force F is removed, the machine will run backward if $R\,dy > Q$. The limiting condition is $R\,dy = Q$, whence $e = \frac{1}{2}$. Therefore the machine will run backward if the efficiency is greater than one half; it is irreversible if its efficiency is less than one half. This rule is not strictly true, since the friction is not entirely independent of the forces transmitted; the friction is usually somewhat less when the machine is run backward than when it is run forward.

EXAMPLE

Find the load W which may be lifted by a force P applied to the Weston differential hoist shown in Fig. 308. Also find the frictional moment necessary to prevent the hoist from running backward under the load.

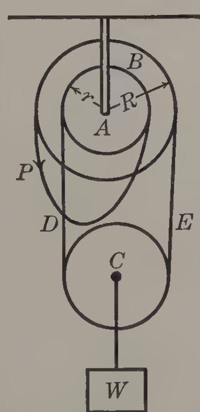

FIG. 308

Solution. The work done by the force P when the upper pulley is turned through an angle $d\theta$ is $PR\,d\theta$. The work that is accomplished in raising the weight W is $\frac{W(R-r)}{2}\,d\theta$. Hence the efficiency is

$$e = \frac{\frac{W(R-r)\,d\theta}{2}}{PR\,d\theta}, \tag{1}$$

from which
$$P = \frac{W(R-r)}{2\,eR}, \tag{2}$$

or
$$W = \frac{2\,ePR}{R-r}.$$

When the force P is not acting, the tension in each of the chains D and E is $\frac{W}{2}$. In order that the weight W may not descend, the frictional moment T of the pulley must not be less than $\frac{WR}{2} - \frac{Wr}{2}$. When the weight W is just on the point of descending,

$$T = \frac{W(R-r)}{2} \tag{3}$$

if the friction at C and the weight of the chain are neglected and the weight of the lower pulley is included in the weight W.

Also, since the work done by the force P is equal to the work done in lifting the weight W, plus the work done in overcoming the frictional couple T,

$$PR\,d\theta = T\,d\theta + \frac{W(R-r)\,d\theta}{2}. \tag{4}$$

Eliminating T between (3) and (4) gives

$$PR = W(R-r),$$

and, from (1),

$$e = \tfrac{1}{2}.$$

PROBLEMS

1. The efficiency of a single pulley is 91 per cent. Find the number of such pulleys required for a self-locking tackle. *Ans.* 8.

2. The larger of the upper pulleys of a Weston hoist is 12 in. in diameter. Find the diameter of the smaller pulley if 100 lb. applied to the chain lifts a load of 900 lb. ($e = 30$ per cent). *Ans.* 11.2 in.

3. In the tackle shown in Fig. 309 it is found that a force of 60 lb. is required to lift a load of 155 lb. Find the efficiency of the tackle. If the friction is 90 per cent of its former value when the motion is reversed, find the force required to lower the same load. *Ans.* 64.6 per cent, 19.6 lb.

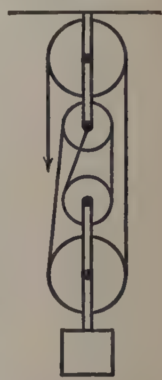

Fig. 309

4. Find the horse power of the motor required to lift an elevator weighing 4400 lb. at a uniform speed of 200 ft. per minute if the driving efficiency is 80 per cent, the worm-gear efficiency is 60 per cent, and the motor efficiency is 80 per cent. *Ans.* 69.4 H.P.

5. A Prony brake has a lever arm 4 ft. long. Find the brake horse power of an engine running at 110 R.P.M. if the net force at the scales is 200 lb. *Ans.* 16.76 H.P.

CHAPTER XVIII

PLANE MOTION OF A RIGID BODY

177. Equations of motion. The motion of the center of gravity of a rigid body is the same as if the total mass of the body were concentrated at its center of gravity and the forces acting upon the body were applied at that point parallel to their original directions (§ 158). Hence the equations of motion of the center of gravity of a body in motion parallel to the xy plane referred to fixed axes in the plane are

$$X = M\frac{d^2\bar{x}}{dt^2}, \tag{1}$$

$$Y = M\frac{d^2\bar{y}}{dt^2}, \tag{2}$$

where X and Y are the x and y components of the external forces acting upon the body, $M = \frac{W}{g}$ is the mass of the body, and $\frac{d^2\bar{x}}{dt^2}$ and $\frac{d^2\bar{y}}{dt^2}$ are the component accelerations of the *center of gravity.*

The motion of a rigid body relative to axes through the center of gravity which remain parallel to fixed axes is the same as if the center of gravity were fixed and the same external forces acted upon the body. The equation of motion of a rigid body about a fixed axis is, by § 162,

$$N = I\frac{d^2\theta}{dt^2}, \tag{3}$$

where N is the moment of all the external forces about an axis passing through the center of gravity perpendicular to the plane of motion, I is the moment of inertia about the axis, and $\frac{d^2\theta}{dt^2}$ is the angular acceleration of the body, the angle θ being measured from any plane through the axis of rotation having its direction fixed in space. There will be as many geometric equations as there are unknown forces acting upon the system.

178. Alternative equations of motion. Any plane motion of a rigid body may be considered as a rotation of the body about

an arbitrary axis perpendicular to the plane of motion together with a translation of the body in the plane of motion. The equations of motion of the body referred to fixed axes may be obtained as follows:

Let the xy plane be the plane of motion, and let (x_0, y_0) be the coördinates referred to fixed axes xoy of an arbitrary point B of the body, a perpendicular through which is the axis of rotation. Also let $(\bar{x}_r, \bar{y}_r)$ or (r, θ) be the coördinates of the center of gravity referred to a set of translating axes x_rBy_r. The x and y components of the acceleration of the center of gravity referred to the fixed axes are (§ 139)

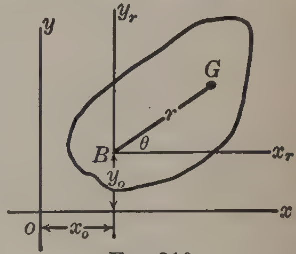

FIG. 310

$$\frac{d^2\bar{x}}{dt^2} = \frac{d^2x_0}{dt^2} + \frac{d^2\bar{x}_r}{dt^2}$$

and
$$\frac{d^2\bar{y}}{dt^2} = \frac{d^2y_0}{dt^2} + \frac{d^2\bar{y}_r}{dt^2},$$

which, by equations (13) and (14) of § 127, become

$$\frac{d^2\bar{x}}{dt^2} = \frac{d^2x_0}{dt^2} - \omega^2\bar{x}_r - \alpha\bar{y}_r$$

and
$$\frac{d^2\bar{y}}{dt^2} = \frac{d^2y_0}{dt^2} - \omega^2\bar{y}_r + \alpha\bar{x}_r.$$

Hence the equations of motion of the center of gravity are, by equations (1) and (2) of § 177,

$$X = M\left(\frac{d^2x_0}{dt^2} - \omega^2\bar{x}_r - \alpha\bar{y}_r\right) \tag{1}$$

and
$$Y = M\left(\frac{d^2y_0}{dt^2} - \omega^2\bar{y}_r + \alpha\bar{x}_r\right). \tag{2}$$

The equation of motion of rotation of the body about the arbitrary axis through B is obtained by equating the sum of the moments of the external forces about B to the moment of the effective forces. The moments of the effective forces about B are

$$-M\left(\frac{d^2x_0}{dt^2} - \omega^2\bar{x}_r - \alpha y_r\right)\bar{y}_r,$$

$$M\left(\frac{d^2y_0}{dt^2} - \omega^2\bar{y}_r + \alpha\bar{x}_r\right)\bar{x}_r,$$

and
$$I_G\alpha.$$

Hence $N_B = M\left(\bar{x}_r \dfrac{d^2y_0}{dt^2} - \bar{y}_r \dfrac{d^2x_0}{dt^2}\right) + (Mr^2 + I_G)\,\alpha,$

or
$$N_B = M\left(\bar{x}_r \frac{d^2y_0}{dt^2} - \bar{y}_r \frac{d^2x_0}{dt^2}\right) + I_B\alpha. \tag{3}$$

179. Choice of axes in taking moments. The moment equation

$$N_B = M\left(\bar{x}_r \frac{d^2y_0}{dt^2} - \bar{y}_r \frac{d^2x_0}{dt^2}\right) + I_B\alpha \tag{1}$$

is valid for any axis perpendicular to the plane of motion. A simpler moment equation,

$$N = I\alpha \tag{2}$$

has been obtained in § 162, where the axis is a fixed axis, and also in § 177, where the axis passes through the center of gravity. It is advantageous from both the standpoint of simplicity and that of correctness to know when (2) may be used, that is, when $M\left(\bar{x}_r \dfrac{d^2y_0}{dt^2} - \bar{y}_r \dfrac{d^2x_0}{dt^2}\right)$ vanishes.

Under any one of the following conditions (1) reduces to the simple form of (2):

(*a*) When
$$\frac{d^2x_0}{dt^2} = \frac{d^2y_0}{dt^2} = 0.$$

Either the axis is fixed in space or it moves without acceleration; that is, it is the point of zero acceleration.

(*b*) When
$$\bar{x}_r = \bar{y}_r = 0.$$

The axis B coincides with the axis through the center of gravity.

(*c*) When
$$\bar{x}_r \frac{d^2y_0}{dt^2} - \bar{y}_r \frac{d^2x_0}{dt^2} = 0,$$

or
$$\frac{\bar{y}_r}{\bar{x}_r} = \frac{\dfrac{d^2y_0}{dt^2}}{\dfrac{d^2x_0}{dt^2}}.$$

The acceleration of the axis B is directed toward the center of gravity; for $\dfrac{\bar{y}_r}{\bar{x}_r}$ is the slope of the line joining B and G, and $\dfrac{\dfrac{d^2y_0}{dt^2}}{\dfrac{d^2x_0}{dt^2}}$ is the slope of the acceleration vector of the axis B.

180. Energy equation of a rigid body. If the equations of motion,

$$X = M \frac{d^2x}{dt^2}, \tag{1}$$

$$Y = M \frac{d^2y}{dt^2}, \tag{2}$$

$$N = I \frac{d^2\theta}{dt^2}, \tag{3}$$

be multiplied by dx, dy, and $d\theta$ respectively, and the results added,

$$M\left(\frac{d^2x}{dt^2}\,dx + \frac{d^2y}{dt^2}\,dy\right) + I\frac{d^2\theta}{dt^2}\,d\theta = X\,dx + Y\,dy + N\,d\theta. \tag{4}$$

Also, if v is the velocity of the center of gravity and ω is the angular velocity of the body,

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2, \tag{5}$$

and

$$\omega^2 = \left(\frac{d\theta}{dt}\right)^2. \tag{6}$$

Differentiating (5) and (6) with respect to t and multiplying by dt,

$$v\,dv = \frac{d^2x}{dt^2}\,dx + \frac{d^2y}{dt^2}\,dy, \tag{7}$$

and

$$\omega\,d\omega = \frac{d^2\theta}{dt^2}\,d\theta. \tag{8}$$

Making proper substitutions from (7) and (8) in (4),

$$Mv\,dv + I\omega\,d\omega = X\,dx + Y\,dy + N\,d\theta. \tag{9}$$

If v_0 and ω_0 are the initial values of v and ω, the integration of (9) between limits gives

$$\frac{Mv^2}{2} - \frac{Mv_0^2}{2} + \frac{I\omega^2}{2} - \frac{I\omega_0^2}{2} = \int(X\,dx + Y\,dy + N\,d\theta). \tag{10}$$

The expression $\frac{Mv^2}{2} - \frac{Mv_0^2}{2}$ is evidently the change in kinetic energy of a particle having the same mass as the body placed at the center of gravity and moving with it, and the expression $\frac{I\omega^2}{2} - \frac{I\omega_0^2}{2}$ is the change in kinetic energy of rotation of the body about a fixed axis through the center of gravity (§ 163).

Also the work done in any displacement of the body by the external forces is, by §§ 96 and 97,

$$\int (X\,dx + Y\,dy + N\,d\theta).$$

Equation (10) may therefore be stated as follows:

The total kinetic energy of a rigid body of mass M moving with plane motion is the sum of (1) *the kinetic energy of a single particle of mass M moving with the center of gravity of the body and* (2) *the kinetic energy of rotation about an axis passing through the center of gravity perpendicular to the plane of motion.*

EXAMPLES

1. A sphere of radius r rolls down a rough plane inclined at an angle ϕ to the horizontal. Find the acceleration of the center of the sphere and the friction on the plane.

Solution. First method. Let K be the normal reaction of the plane, and let F be the friction. Let x be measured positively downward from the point O occupied by the center of the sphere in its initial position, and let θ be the angle between a radius fixed in the sphere and a fixed line in space. The equations of motion are (§ 177)

$$W \sin\phi - F = \frac{W}{g}\frac{d^2x}{dt^2}, \qquad (1)$$

$$W \cos\phi - K = \frac{W}{g}\frac{d^2y}{dt^2}, \qquad (2)$$

$$Fr = \frac{2}{5}\frac{W}{g}r^2\frac{d^2\theta}{dt^2}, \qquad (3)$$

and the geometric equation is

$$x = r\theta, \qquad (4)$$

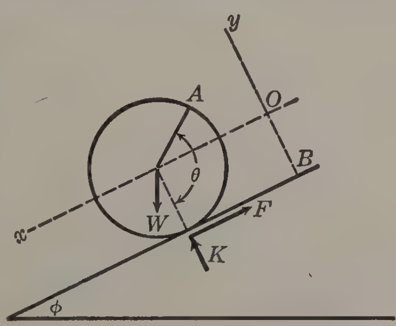

FIG. 311

if the points A and B are coincident at the beginning of the motion.

Differentiating (4), $$\frac{d^2x}{dt^2} = r\frac{d^2\theta}{dt^2}. \qquad (5)$$

Since $\frac{d^2y}{dt^2} = 0$, (2) gives $K = W\cos\phi.$

Eliminating $\frac{d^2x}{dt^2}$ and $\frac{d^2\theta}{dt^2}$ between (1), (3), and (5),

$$F = \frac{2}{7}W\sin\phi. \qquad (6)$$

Hence, from (1), $$\frac{d^2x}{dt^2} = \frac{5}{7}g\sin\phi. \qquad (7)$$

Energy method. The work of gravity when the sphere descends a distance x is $Wx \sin \phi$. Also, by § 180, the total kinetic energy of the sphere is

$$\frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} \frac{W}{g} v^2 + \frac{1}{2}\left(\frac{2}{5} \frac{W}{g} r^2\right)\left(\frac{d\theta}{dt}\right)^2. \tag{1}$$

But, since $x = r\theta$, $$\left(\frac{d\theta}{dt}\right)^2 = \frac{v^2}{r^2},$$

and hence $$\frac{1}{2} \frac{W}{g} v^2 \left(1 + \frac{2}{5}\right) = Wx \sin \phi, \tag{2}$$

from which $$v^2 = \frac{10}{7} gx \sin \phi. \tag{3}$$

Also, since the acceleration is constant, from ((7) § 143),

$$v^2 = 2\,x \frac{d^2x}{dt^2}, \tag{4}$$

which, substituted in (3), gives

$$\frac{d^2x}{dt^2} = \frac{5}{7} g \sin \phi. \tag{5}$$

Third method. By § 179 the moment center may be taken at any point as though that point were fixed, if the acceleration of the point passes through the center of gravity. The instantaneous center of rotation of a rolling sphere fulfills this condition (Problem 12, p. 201). Therefore, taking moments about the point where the sphere touches the plane,

$$Wr \sin \phi = \frac{7}{5} \frac{W}{g} r^2 \frac{d^2\theta}{dt^2}. \tag{1}$$

Combining (1) with (5) of the first solution,

$$\frac{d^2x}{dt^2} = \frac{5}{7} g \sin \phi. \tag{2}$$

2. A sphere of radius r partly rolls and partly slides down an inclined plane making an angle ϕ with the horizontal. Find the acceleration of the center of the sphere and its angular acceleration.

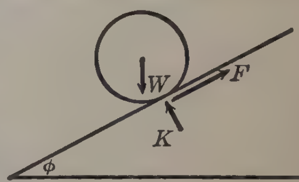

FIG. 312

Solution. If friction F is not great enough to cause rolling without slipping, the geometric equation $x = r\theta$ is no longer valid and the equations of motion are (see Example 1)

$$Fr = \frac{2}{5} \frac{W}{g} r^2 \frac{d^2\theta}{dt^2}, \tag{1}$$

$$W \sin \phi - F = \frac{W}{g} \frac{d^2x}{dt^2}. \tag{2}$$

Also, since $K = W \cos \phi$ and $F = \mu K$,

$$F = \mu W \cos \phi, \tag{3}$$

and $$\frac{d^2\theta}{dt^2} = \frac{5\,\mu g \cos \phi}{2\,r}. \tag{4}$$

Integrating, $$\frac{d\theta}{dt} = \frac{5\,\mu g \cos\phi}{2\,r}\,t + C; \tag{5}$$

and since $\frac{d\theta}{dt} = 0$ when $t = 0$, $C = 0$.

Integrating again, $$\theta = \frac{5\,\mu g \cos\phi}{4\,r}\,t^2, \tag{6}$$

the constant of integration being zero.

Equations (2) and (3) give

$$\frac{d^2x}{dt^2} = g\,(\sin\phi - \mu\cos\phi). \tag{7}$$

Integrating, $$\frac{dx}{dt} = gt\,(\sin\phi - \mu\cos\phi), \tag{8}$$

and $$x = \frac{gt^2}{2}\,(\sin\phi - \mu\cos\phi). \tag{9}$$

The constants of integration vanish as before. In the case where the sphere partly slides and partly rolls there is a loss of energy in overcoming the force of friction. There is no loss of energy if the motion is pure rolling or pure sliding.

PROBLEMS

1. A thin hoop of radius r rolls without slipping down an inclined plane making an angle ϕ with the horizontal. Find the acceleration of its center. *Ans.* $\frac{g\sin\phi}{2}$.

2. The hoop of Problem 1 is replaced by a solid cylinder of radius r. Find the acceleration of the center of the cylinder. *Ans.* $\frac{2}{3}\,g\sin\phi$.

3. A hollow cylinder whose inner radius is one half the outer radius rolls without slipping down an inclined plane making an angle ϕ with the horizontal. Find the acceleration of the center of the cylinder. *Ans.* $\frac{8}{13}\,g\sin\phi$.

4. If the inclination of the plane of Problem 3 is so great as to cause the hollow cylinder to partly slip and partly roll, find the distance it will move down the plane in time t if it starts from rest.

Ans. $\frac{gt^2}{2}\,(\sin\phi - \mu\cos\phi)$.

5. A hollow sphere of inner radius one half the outer radius rolls without slipping down an inclined plane making an angle ϕ with the horizontal. Find the acceleration. *Ans.* $\frac{70}{101}\,g\sin\phi$.

6. A string which is wrapped around a solid cylinder of weight W and radius r has one end attached to a ceiling. If the cylinder falls in a vertical direction with its axis horizontal, find the tension in the string and the acceleration of the center of the cylinder. Note that this is a special case of a cylinder rolling down an inclined plane.

Ans. $\frac{W}{3}$, $\frac{2}{3}\,g$.

7. A circular disk whose diameter is 2 ft. and whose weight is 10 lb. is supported by two concentric disks attached to it, each having a diameter of 1 ft. and a weight of 2 lb., which roll on rough horizontal rails. A string wound about the large disk passes over a smooth peg and supports a weight of 8 lb. at its end, as shown in Fig. 313. Find the tension in the string and the acceleration of the center of the cylinder. *Ans.* 2.67 lb., 7.16 ft. per second per second.

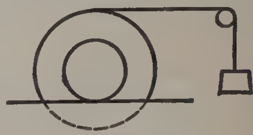

FIG. 313

8. A wheel weighing 3000 lb. has a diameter of 6 ft. The center of gravity of the wheel is eccentric a distance of 6 in. If the wheel is placed on a horizontal plane with the center of gravity vertically above the geometric center and is given a slight displacement, find the total vertical reaction on the plane when the center of gravity has been displaced to its lowest position. The radius of gyration of the wheel with respect to the center of gravity is 2 ft. (Assume that rolling friction is neglected.) *Ans.* 3293 lb.

9. A spool weighing 10 lb., having an outer radius of 3 ft., an inner radius of 1 ft., and a radius of gyration of 2 ft. about its center, is placed on a perfectly rough inclined plane making an angle of 30° with the horizontal. A string wound about the central section of the spool is led off from its under side parallel to the plane and thence up over a smooth peg to a suspended weight of 20 lb. Find the acceleration of the center of the spool and the tension in the cord. *Ans.* 11.49 ft. per second per second, 15.2 lb.

10. A wagon consists of four wheels in the form of solid disks 1 ft. in diameter, weighing 8 lb. each, and a running gear and box, weighing 28 lb. Find the acceleration of the wagon down a rough incline making an angle of 30° with the horizontal, axle friction being neglected.

HINT. Use the energy equation together with $v^2 = 2\,as$.

Ans. $a = 12.7$ ft. per second per second.

11. A wheel and axle B weighs 24 lb., has a radius of gyration of 2 ft., and is arranged so that as the weight A descends the axle rolls on the vertical cord attached to the ceiling at C. Find (*a*) the tension in each cord, (*b*) the acceleration of the center E, and (*c*) the length of cord unwound from the wheel in 2 sec. *Ans.* (*a*) 12.8 lb., 39.7 lb.; (*b*) $a = \frac{3}{25}\,g$; (*c*) 30.91 ft.

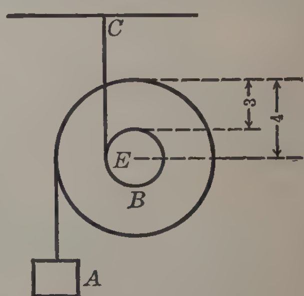

FIG. 314

EXAMPLES

1. A uniform rod of length l and weight W is placed in a vertical position with its lower end on a rough floor. The upper end being slightly displaced, determine the vertical reaction and the force of friction at the lower end at any angle θ with the vertical.

Solution. Let K and F be the required forces at the lower end of the rod, and let (x, y) be the coördinates of the center of the rod. Since the motion of the center of gravity is the same as if the forces acting on the rod were translated parallel to themselves to act at the center of gravity,

$$F = \frac{W}{g}\frac{d^2x}{dt^2}, \qquad (1)$$

and

$$K - W = \frac{W}{g}\frac{d^2y}{dt^2}. \qquad (2)$$

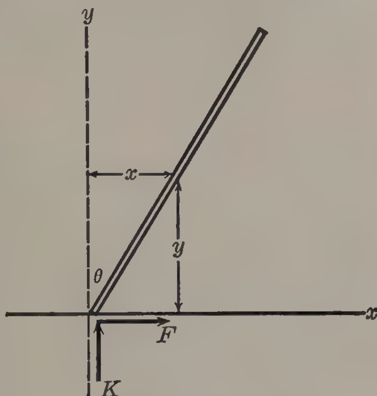

FIG. 315

From the figure, $x = \frac{l}{2}\sin\theta$ and $y = \frac{l}{2}\cos\theta$, and, by differentiation,

$$\frac{d^2x}{dt^2} = \frac{l}{2}\cos\theta\frac{d^2\theta}{dt^2} - \frac{l}{2}\sin\theta\left(\frac{d\theta}{dt}\right)^2, \qquad (3)$$

and

$$\frac{d^2y}{dt^2} = -\frac{l}{2}\sin\theta\frac{d^2\theta}{dt^2} - \frac{l}{2}\cos\theta\left(\frac{d\theta}{dt}\right)^2. \qquad (4)$$

Eliminating $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ between (1), (2), (3), and (4),

$$F = \frac{Wl}{2\,g}\left[\cos\theta\frac{d^2\theta}{dt^2} - \sin\theta\left(\frac{d\theta}{dt}\right)^2\right], \qquad (5)$$

and

$$K - W = -\frac{Wl}{2\,g}\left[\sin\theta\frac{d^2\theta}{dt^2} + \cos\theta\left(\frac{d\theta}{dt}\right)^2\right]. \qquad (6)$$

Also, since the motion of rotation is the same as if the center of gravity were fixed (§ 160), the moment equation about the center of gravity is

$$\frac{Kl}{2}\sin\theta - \frac{Fl}{2}\cos\theta = \frac{1}{12}\frac{W}{g}l^2\frac{d^2\theta}{dt^2}. \qquad (7)$$

Substituting the values of K and F from (5) and (6) in (7),

$$\frac{d^2\theta}{dt^2} = \frac{3}{2}\frac{g}{l}\sin\theta. \qquad (8)$$

Multiplying both sides of (8) by $\frac{d\theta}{dt}$ and integrating,

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = -\frac{3}{2}\frac{g}{l}\cos\theta + C. \qquad (9)$$

Also, since $\frac{d\theta}{dt} = 0$ when $\theta = 0$, $C = \frac{3g}{2l}$

and
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{l}(1 - \cos\theta). \tag{10}$$

Equation (10) could have been written at once from the energy equation.

Substituting the values of $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ from (10) and (8) in (5) and (6),

$$F = \frac{3}{2} W \sin\theta\left(\frac{3}{2}\cos\theta - 1\right), \tag{11}$$

and
$$K = \frac{W}{4}(1 - 3\cos\theta)^2. \tag{12}$$

The vertical reaction K becomes zero when $\cos\theta = \frac{1}{3}$, but it does not change sign, and therefore the end of the rod does not leave the floor. The force of friction F changes sign when θ passes through the value $\cos^{-1}\frac{2}{3}$, so that the rod will begin to slide for some value of θ between zero and $\cos^{-1}\frac{1}{3}$ unless the end of the rod is hinged to the floor. The changes of F and K as θ varies are best exhibited by plotting (11) and (12).

2. The length of the connecting rod of a steam engine is l, the length of the crank is r, and θ is the crank angle, Fig. 316. It is required to find the components Q and T of the crank-pin reactions and the reaction R at the guides for any crank angle θ. The engine runs at a constant angular velocity of n revolutions per minute. The force between the piston rod and the crosshead is represented by P.

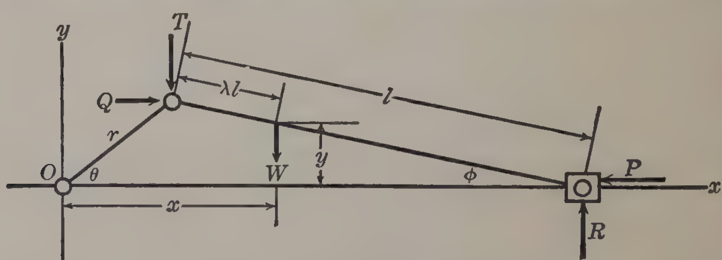

FIG. 316

Solution. Let the axes be taken as shown in Fig. 316, and let the center of gravity of the connecting rod be at a distance λl from the crank pin. Since the motion of the center of gravity is the same as if all the external forces were translated parallel to themselves to act at that point, two of the equations of motion of the connecting rod are

$$Q - P = \frac{W}{g}\frac{d^2x}{dt^2} \tag{1}$$

and
$$R - T - W = \frac{W}{g}\frac{d^2y}{dt^2}, \tag{2}$$

where W is the weight of the connecting rod. Taking moments about the center of gravity,

$$Pl(1-\lambda)\sin\phi - T\lambda l\cos\phi - Rl(1-\lambda)\cos\phi + Q\lambda l\sin\phi = \frac{W}{g}k^2\frac{d^2\phi}{dt^2}. \quad (3)$$

The coördinates of the center of gravity are

$$x = r\cos\theta + \lambda l\cos\phi \quad (4)$$

and
$$y = r\sin\theta - \lambda l\sin\phi. \quad (5)$$

Also
$$r\sin\theta = l\sin\phi, \quad (6)$$

and hence
$$\sin\phi = \frac{r}{l}\sin\theta, \quad (7)$$

and
$$\cos\phi = \frac{\sqrt{l^2 - r^2\sin^2\theta}}{l}. \quad (8)$$

Substituting the values of $\sin\phi$ and $\cos\phi$ from (7) and (8) in (4) and (5) and differentiating,

$$\frac{d^2x}{dt^2} = -r\left(\frac{d\theta}{dt}\right)^2\left\{\cos\theta + \lambda r\left[\frac{l^2\cos 2\theta + r^2\sin^4\theta}{(l^2 - r^2\sin^2\theta)^{\frac{3}{2}}}\right]\right\}, \quad (9)$$

and
$$\frac{d^2y}{dt^2} = -r(1-\lambda)\sin\theta\left(\frac{d\theta}{dt}\right)^2. \quad (10)$$

The terms involving $\frac{d^2\theta}{dt^2}$ vanish, since the angular velocity of the crank is constant.

The angular acceleration of the connecting rod is found by differentiating (7) twice, giving

$$\frac{d^2\phi}{dt^2} = \frac{-r\sin\theta(l^2 - r^2)\left(\frac{d\theta}{dt}\right)^2}{(l^2 - r^2\sin^2\theta)^{\frac{3}{2}}}. \quad (11)$$

Evidently $\frac{d\theta}{dt} = \frac{n\pi}{30}$. (12)

In a particular case the values of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, and $\frac{d^2\phi}{dt^2}$ are calculated from (9), (10), and (11) respectively and substituted in (1), (2), and (3). The unknown reactions T, Q, and R are then obtained by solving (1), (2), and (3) simultaneously. Since friction is neglected, the required twisting moment or torque N is given by

$$N = Qr\sin\theta + Tr\cos\theta. \quad (13)$$

3. Find the thrust P of the piston rod on the connecting rod of Example 2 corresponding to any crank angle θ, having given the pressure p per square inch on the piston, the area A of the piston, and the weight W of the piston, piston rod, and crosshead.

Solution. Let x be the distance from the origin O to the crosshead end of the connecting rod, Fig. 316. Then

$$x = r\cos\theta + l\cos\phi. \quad (1)$$

FIG. 317

The value of x from (1) is the same as the value of x from (4), Example 2, if λ is made unity. Hence the value of $\frac{d^2x}{dt^2}$ is

$$\frac{d^2x}{dt^2} = -r\left(\frac{d\theta}{dt}\right)^2\left\{\cos\theta + r\left[\frac{l^2\cos 2\theta + r^2\sin^4\theta}{(l^2 - r^2\sin^2\theta)^{\frac{3}{2}}}\right]\right\} \tag{2}$$

from (9), Example 2.

The total pressure on the piston is pA, where p is the difference in pressure per unit area between the steam pressure on one side of the piston and the back pressure on the other as obtained from the indicator card. The force causing acceleration of the connecting rod in the x direction is $P - pA$, and hence

$$P - pA = \frac{W}{g}\frac{d^2x}{dt^2}. \tag{3}$$

The force P is found by substituting the value of $\frac{d^2x}{dt^2}$ from (2) in (3). In the case of a vertical engine, the weight W must also be included in (3) as one of the external forces. The force of friction is neglected.

4. Make use of the equations of Examples 2 and 3 to find the twisting moment of a horizontal steam engine having a connecting rod 6 ft. long weighing 300 lb. The crank is 2 ft. long. The center of gravity of the connecting rod is 4 ft. from the crosshead end, and its radius of gyration about the center of gravity is 2 ft. The engine runs at 240 R.P.M., and the crank angle is 30° with the inner dead center as the piston moves toward the crank shaft. The steam pressure on the forward end of the piston is 100 lb. per square inch and on the rear end 10 lb. per square inch. The weight of the piston, piston rod, and crosshead is 1000 lb., and the diameter of the piston is 30 in.

Solution. The angular velocity is

$$\frac{d\theta}{dt} = \frac{240}{60} \times 2\,\pi = 8\,\pi \text{ radians per second.}$$

The acceleration of the piston is, by Example 3,

$$-\,2(8\,\pi)^2\left[0.866 + 2\,\frac{18.25}{(35)^{\frac{3}{2}}}\right] = -\,1310 \text{ ft. per second per second.}$$

The steam pressure on the piston is

$$(100 - 10)\pi(15)^2 = 63{,}617 \text{ lb.}$$

Hence, by (3), Example 3,

$$P = 63{,}617 - \frac{1000}{32.2} \times 1310 = 22{,}934 \text{ lb.}$$

The x component of the acceleration of the center of gravity of the connecting rod is, by (9), Example 2,

$$\frac{d^2x}{dt^2} = -\,2(8\,\pi)^2\left[0.866 + \frac{2}{3}\,\frac{18.25}{(35)^{\frac{3}{2}}}\right] = -\,1169 \text{ ft. per second per second.}$$

Also, by (10), Example 2,

$$\frac{d^2y}{dt^2} = -\frac{2}{6}(4) \times \frac{1}{2} \times (8\,\pi)^2 = -\,421 \text{ ft. per second per second.}$$

The angular acceleration of the connecting rod is, by (11), Example 2,

$$\frac{d^2\phi}{dt^2} = -\frac{2 \times \frac{1}{2} \times 32 \times (8\,\pi)^2}{(35)^{\frac{3}{2}}} = -\,97.6 \text{ radians per second per second.}$$

Substituting in (1), (2), and (3) of Example 2,

$$Q = 12{,}070 \text{ lb.},$$

$$T = 6291 \text{ lb.},$$

$$R = 2667 \text{ lb.},$$

$$N = 22{,}966 \text{ lb.-ft.}$$

PROBLEMS

1. Where extreme accuracy is not required in steam-engine problems, it is usual to neglect the square and higher powers of the sine of the crank angle. Solve Example 4, p. 290, neglecting the squares and higher powers of the sine of θ and compare the results with the accurate method.

Ans. $\frac{d^2x}{dt^2}$ of piston $= -\,1305$ ft. per second per second,
$P = 23{,}090$ lb.,
$\frac{d^2x}{dt^2}$ of rod $= -\,1163$ ft. per second per second,
$\frac{d^2y}{dt^2}$ of rod $= -\,421$ ft. per second per second,
$\frac{d^2\phi}{dt^2}$ of rod $= -\,93.6$ radians per second per second,
$Q = 12{,}255$ lb., $T = 6290$ lb.,
$N = 23{,}143$ lb.-ft., $R = 2670$ lb.

2. A uniform rod is placed in a vertical plane with one end resting on a smooth horizontal plane and the other end in contact with a smooth vertical plane. Find its inclination to the floor when it leaves the wall if it starts from rest in the vertical position.

HINT. The rod leaves the wall when the reaction at the wall changes sign.

Ans. $\sin\theta = \frac{2}{3}$.

3. A uniform rod is placed with one end in contact with a rough horizontal plane, the angle between the rod and the plane being 60°. Find the coefficient of friction so that it will be on the point of sliding the instant it is released.

Ans. $\frac{3\sqrt{3}}{13}$.

4. Two equal rods, of length $2\,a$, joined together at their centers by a pivot to form an X, are placed in a vertical plane with their lower ends in contact with a perfectly smooth plane. If the initial vertical angle between the rods is $2\,\alpha$, find the velocity of the joint just as it comes in contact with the horizontal plane. Would the result be the same if the plane were rough but not too rough to prevent sliding?

Ans. $\sqrt{\dfrac{3\,ga\cos\alpha}{2}}$.

5. A rod AB, of weight W and length $2\,a$, moves in a vertical plane with its ends in contact with a smooth vertical wall and a smooth horizontal floor. The center of the rod C is jointed to a rod CD, of length a and weight $\frac{W}{2}$, which in turn is jointed to the intersection of the wall and floor. Find the angular velocity of the rod AB just as it touches the floor if its initial inclination to the floor is α.

Ans. $\sqrt{\dfrac{5\,g\sin\alpha}{3\,a}}$.

6. Show that equation (4) of Example 2, p. 289, may be written

$$x = r\cos\theta + \lambda l\sqrt{1 - \frac{r^2}{l^2}\sin^2\theta},$$

or

$$x = r\cos\theta + \lambda l\left(1 - \frac{1}{2}\frac{r^2}{l^2}\sin^2\theta - \frac{1}{8}\frac{r^4}{l^4}\sin^4\theta\cdots\right).$$

Neglecting higher powers of $\sin\theta$, show that

$$\frac{dx}{dt} = -r\frac{d\theta}{dt}\left[\sin\theta + \frac{\lambda r}{2\,l}\sin 2\,\theta\right]$$

and $$\frac{d^2x}{dt^2} = -r\left(\frac{d\theta}{dt}\right)^2\left(\cos\theta + \frac{\lambda r}{l}\cos 2\,\theta\right) - r\frac{d^2\theta}{dt^2}\left(\sin\theta + \frac{\lambda r}{2\,l}\sin 2\,\theta\right).$$

7. A spool A, weighing 64.4 lb., rolls on a rough horizontal plane under the action of a rough string which passes over a smooth peg D, under the rough pulley B, and thence to the point E, where it is attached to the ceiling. The outer radius, the inner radius, and the radius of gyration of the spool A are 3 ft., 1 ft., and 2 ft. respectively. The radius of the pulley B is 2 ft., its radius of gyration is 1 ft., and it weighs 32.2 lb. If C weighs 32.2 lb., find its acceleration. *Ans.* 7.36 ft. per second per second.

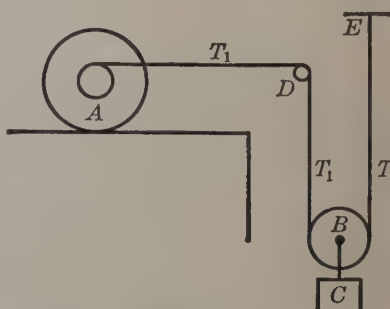

FIG. 318

8. A car is free to roll on a straight level track. A heavy wheel is mounted on the car with its axle transverse to the car. Compare the acceleration of the car (*a*) when a force P is applied to the car parallel to the track and (*b*) when the same force P is applied to the heavy wheel by means of a rope wound on the wheel.

9. A sled weighing 50 lb. carries a cylinder of the same weight mounted on a horizontal axle transverse to the sled. The radius of the cylinder is 3 ft. and its radius of gyration is 2 ft. A rope wound on the cylinder is led off horizontally and perpendicularly to the axle, over a smooth fixed peg, and thence vertically to a 100-pound weight. Find the acceleration of the sled, friction being neglected.

Ans. $\frac{2g}{13}$ ft. per second per second.

EXAMPLE

A solid cylinder of radius r rolls down a fixed cylinder of radius R, which is rough enough to prevent slipping. If the first cylinder starts from rest in the position of unstable equilibrium, find the position where it will leave the second cylinder.

Solution. Let C be the point of the rolling cylinder which was originally at A. Since no slipping occurs,

$$R\theta = r\phi. \tag{1}$$

Let β be the angle which the radius CD makes with a fixed axis of reference. Then, by geometry,

$$\beta = \theta + \phi, \tag{2}$$

and

$$\frac{d^2\beta}{dt^2} = \frac{d^2\theta}{dt^2} + \frac{d^2\phi}{dt^2}. \tag{3}$$

Also

$$R\frac{d^2\theta}{dt^2} = r\frac{d^2\phi}{dt^2}, \tag{4}$$

and hence

$$\frac{d^2\beta}{dt^2} = \frac{d^2\theta}{dt^2}\left(1 + \frac{R}{r}\right). \tag{5}$$

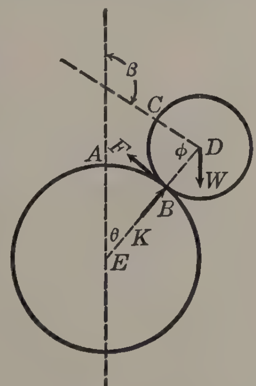

FIG. 319

The acceleration of the point of contact B is directed through the center of gravity of the moving cylinder (Problem 3, p. 209), and hence moments may be taken at the point B as if it were a fixed point. Therefore

$$Wr\sin\theta = \frac{3}{2}\frac{W}{g}r^2\frac{d^2\beta}{dt^2} = \frac{3}{2}\frac{W}{g}r^2\left(\frac{r+R}{r}\right)\frac{d^2\theta}{dt^2}. \tag{6}$$

Equation (6) may be written

$$\frac{d^2\theta}{dt^2} = \frac{2g\sin\theta}{3(R+r)}. \tag{7}$$

Multiplying both sides of (7) by $\frac{d\theta}{dt}$ and integrating,

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 + C = \frac{-2g\cos\theta}{3(R+r)}. \tag{8}$$

Also, since $\theta = 0$ when $\frac{d\theta}{dt} = 0$,

$$C = \frac{-2g}{3(R+r)},$$

and

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{4g}{3(R+r)}(1 - \cos\theta). \tag{9}$$

This equation might also have been written at once from the equation of energy. Since the motion of the center of gravity of the cylinder is the same as if the normal reaction K and the friction F acted at that point, the equations of motion are

$$W \sin\theta - F = \frac{W}{g}(R+r)\frac{d^2\theta}{dt^2}, \tag{10}$$

and

$$W \cos\theta - K = \frac{W}{g}(R+r)\left(\frac{d\theta}{dt}\right)^2. \tag{11}$$

Substituting the values of $\frac{d^2\theta}{dt^2}$ and $\left(\frac{d\theta}{dt}\right)^2$ from (7) and (9) in (10) and (11),

$$F = \frac{W}{3}\sin\theta \quad \text{and} \quad K = \frac{W}{3}(7\cos\theta - 4). \tag{12}$$

The value of K therefore changes sign when θ becomes larger than $\cos^{-1}\frac{4}{7}$, and the upper cylinder leaves the lower cylinder.

PROBLEMS

1. Derive equation (9) of the example above by using the equation of energy.

2. Let the cylinders of the example on page 293 be imperfectly rough, and let μ be the coefficient of friction. Show that slipping will occur when the angle θ, Fig. 319, is given by the equation

$$\mu(7\cos\theta - 4) = \sin\theta.$$

3. A perfectly rough sphere rolls down a fixed sphere, starting from rest in the position of unstable equilibrium. If θ is the angle between the common normal and the vertical, show that the spheres will part when $\cos\theta = \frac{10}{17}$. Does the result depend upon the relative sizes of the spheres?

4. A perfectly rough solid sphere of radius r rolls on the inside of a fixed hollow sphere of radius $4r$. Find the least velocity and the normal reaction of the inner sphere in its lowest position so that it may retain contact with the highest point of the hollow sphere.

Ans. $V^2 = \frac{81}{7}gr$, $\frac{34}{7}W$.

5. A thin hoop of radius r and weight W, rotating with angular velocity ω, is placed on a horizontal floor with its center of gravity at rest and vertically above the point of contact. Find the length of time that slipping occurs, and find the angular velocity after pure rolling begins if μ is the coefficient of friction.

Ans. $\frac{r\omega}{2\mu g}$, $\frac{\omega}{2}$.

CHAPTER XIX

IMPULSE AND MOMENTUM

181. Momentum. The linear momentum of a particle of mass m moving with velocity v is a vector having the same direction and sense as the velocity vector and having a magnitude mv equal to the product of the mass and the velocity. If $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ are the projections of the velocity of a particle along the x, y, and z axes respectively, the projections of the linear momentum of the particle in the same directions are

$$m\frac{dx}{dt}, \quad m\frac{dy}{dt}, \quad \text{and} \quad m\frac{dz}{dt}. \tag{1}$$

The general equations of motion of a rigid body, (7), (8), and (9) of § 157, may be written

$$\Sigma X = \frac{d}{dt}\left(\Sigma m\frac{dx}{dt}\right), \; \Sigma Y = \frac{d}{dt}\left(\Sigma m\frac{dy}{dt}\right), \; \Sigma Z = \frac{d}{dt}\left(\Sigma m\frac{dz}{dt}\right). \tag{2}$$

These equations may be put into more usable form by the introduction of the coördinates of the center of gravity of the body. Let M be the mass of the body and $(\bar{x}, \bar{y}, \bar{z})$ be the coördinates of its center of gravity; then, since

$$\Sigma mx = M\bar{x}, \quad \Sigma my = M\bar{y}, \quad \text{and} \quad \Sigma mz = M\bar{z},$$

therefore

$$\Sigma m\frac{dx}{dt} = M\frac{d\bar{x}}{dt}, \quad \Sigma m\frac{dy}{dt} = M\frac{d\bar{y}}{dt}, \quad \text{and} \quad \Sigma m\frac{dz}{dt} = M\frac{d\bar{z}}{dt}. \tag{3}$$

Making proper substitutions in equations (2), they become

$$\Sigma X = \frac{d}{dt}\left(M\frac{d\bar{x}}{dt}\right), \quad \Sigma Y = \frac{d}{dt}\left(M\frac{d\bar{y}}{dt}\right), \quad \Sigma Z = \frac{d}{dt}\left(M\frac{d\bar{z}}{dt}\right). \tag{4}$$

Equations (3) state that *the linear momentum of a rigid body in any fixed direction is equal to the product of the total mass of*

the body and the component of the velocity of its center of gravity in that direction.

Equations (4) may be expressed in words as follows: *The time rate of change of the momentum of a rigid body in any fixed direction is equal to the sum of the components of the external forces in that direction.*

182. Moment of momentum. The moment of a force about a point or about an axis has been defined in § 22. Similarly the moment of momentum of a particle about an axis is the moment of the linear-momentum vector about that axis. Let P represent the momentum vector mv of a particle and AB any axis. Also let C be any plane perpendicular to the axis, cutting the axis at O, and let P' be the projection of the vector P upon the plane C. Then the moment of momentum of the vector P about the axis AB is the product $P'p$, where p is the perpendicular distance from the point O to the vector P'.

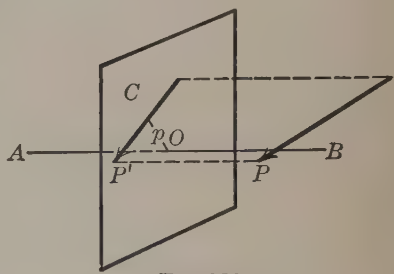

FIG. 320

The moment of momentum of a body with respect to the three fixed axes of reference may be found as follows: Let the momentum mv of any particle A at the point (x, y, z) be represented by the vector AP. The components of the momentum vector AP parallel to the axes of reference are

$$m\frac{dx}{dt},\quad m\frac{dy}{dt},\quad \text{and}\quad m\frac{dz}{dt}$$

respectively. The moments of these momentum vectors about the x, y, and z axes are, by § 55,

$$m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right),\quad m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right),\quad \text{and}\quad m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right) \quad (1)$$

respectively. The moments of momentum of the body about the x, y, and z axes are

$$\Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right),\quad \Sigma m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right),\ \text{and}\ \Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right) \quad (2)$$

respectively.

The general equations of motion, (13), (14), and (15) of § 157, may be written

$$\Sigma(Zy - Yz) = \frac{d}{dt}\left[\Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)\right], \tag{3}$$

$$\Sigma(Xz - Zx) = \frac{d}{dt}\left[\Sigma m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right)\right], \tag{4}$$

$$\Sigma(Yx - Xy) = \frac{d}{dt}\left[\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\right]. \tag{5}$$

These equations show that *the time rate of change of the moment of momentum of a body about a fixed axis is equal to the sum of the moments of the external forces about the axis.*

183. Impulse; impulsive force. Let a particle of mass m move on a straight line under the action of a force F, which may be variable or constant. The fundamental equation of motion, $F = ma$, may be written

$$F = m\frac{dv}{dt}. \tag{1}$$

Let the force F act upon the particle during the time interval $t - t_0$, and let v and v_0 be the velocities of the particle just before and just after the action of the force. Integrating (1),

$$\int_{t_0}^{t} F\,dt = mv - mv_0. \tag{2}$$

The integral $\int_{t_0}^{t} F\,dt$ is called the *impulse* of the force F in the time interval $t - t_0$. Equation (2) shows that the impulse of a force acting on a particle is equal to the change in momentum of the particle. The impulse may be directly evaluated if F is a known function of t; if F is constant, the impulse is $F(t - t_0)$. The impulse may be indirectly evaluated from the change in momentum. The integral $\int_{t_0}^{t} F\,dt$ is sometimes designated by F_i.

A very large force which acts for a very short interval of time is called an *impulsive force*. Such forces arise when bodies which are nearly rigid collide, as, for example, when an anvil is struck by a hammer. A change in momentum of a body may be produced by either an impulsive force or an ordinary force. An impulsive force changes the velocity of a body without changing its position on account of the great magnitude of the force

and the very short interval of time during which it acts. An ordinary force changes the velocity of a body, not, however, without changing its position. When impulsive forces and ordinary forces act simultaneously, the ordinary forces may usually be neglected on account of their relatively small magnitude.

For the case of a rigid body of mass M, whose center of gravity has component velocities (u, v, w) and (u_0, v_0, w_0) at times t and t_0 respectively, equations (4) of § 181 may be integrated, giving

$$\left.\begin{aligned} \Sigma\int_{t_0}^{t} X\,dt &= \left[M\frac{d\bar{x}}{dt}\right]_{u_0}^{u} = M(u-u_0), \\ \Sigma\int_{t_0}^{t} Y\,dt &= \left[M\frac{d\bar{y}}{dt}\right]_{v_0}^{v} = M(v-v_0), \\ \Sigma\int_{t_0}^{t} Z\,dt &= \left[M\frac{d\bar{z}}{dt}\right]_{w_0}^{w} = M(w-w_0). \end{aligned}\right\} \tag{3}$$

184. Moment of an impulse. In case the external forces are impulsive the general equations (3), (4), and (5), § 182, may be modified as follows:

Let u_0 and u, v_0 and v, and w_0 and w be the velocities of a particle of mass m of a body parallel to the x, y, and z axes just before and just after the action of the impulsive forces, and let $t - t_0$ be the time interval of the action of the impulsive forces. Then, since the coördinates x, y, and z may be assumed constant during this interval, the integration of equations (3), (4), and (5), § 182, gives

$$\Sigma\left[y\int_{t_0}^{t} Z\,dt - z\int_{t_0}^{t} Y\,dt\right] = \Sigma m[y(w-w_0) - z(v-v_0)], \tag{1}$$

$$\Sigma\left[z\int_{t_0}^{t} X\,dt - x\int_{t_0}^{t} Z\,dt\right] = \Sigma m[z(u-u_0) - x(w-w_0)], \tag{2}$$

$$\Sigma\left[x\int_{t_0}^{t} Y\,dt - y\int_{t_0}^{t} X\,dt\right] = \Sigma m[x(v-v_0) - y(u-u_0)]. \tag{3}$$

Equations (1), (2), and (3) may be expressed in words as follows: *The change in the moment of momentum of a body about a fixed axis is equal to the sum of the moments of the impulses with respect to that axis.*

EXAMPLES

1. A block of steel weighing 5 lb. is moving on a horizontal plane with a speed of 10 ft. per second when it is struck a sudden blow in its direction of motion by a hammer. Find the impulse if the velocity after the action of the impulsive force is 20 ft. per second in the same direction.

Solution. The impulse is equal to the change in momentum.

Hence the impulse is $\frac{5}{32.2}(20 - 10) = \frac{50}{32.2} = 1.55$ lb.-sec.

2. A constant horizontal force of 500,000 lb. acts upon a body weighing 10 lb. during a time interval of 0.00005 sec. Find the velocity of the body if it was initially at rest on a horizontal surface.

Solution. The impulse of the force is $\int_0^{.00005} 500{,}000\, dt = 25$ lb.-sec., and since the impulse is equal to the change in momentum of the body, the final velocity is found from the equation

$$25 = \frac{10}{32.2}(v - 0).$$

Hence $$v = 80.5 \text{ ft. per second.}$$

3. A stream of water 1 in. in diameter, having a velocity of 100 ft. per second, impinges normally on a stationary plate, thereby losing all its normal velocity. Find the force which the stream exerts upon the plate.

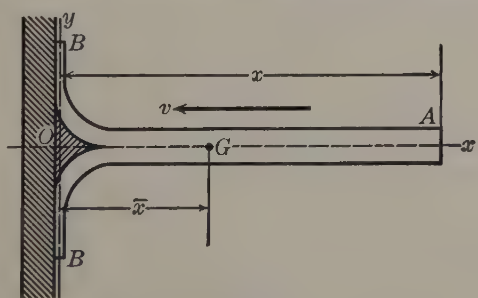

FIG. 321

Solution. Let any length l of the stream be in the position shown in Fig. 321, where x is the length of the portion which has not impinged upon the plate. Let a be the area of the cross section of the stream and w the weight per unit volume. The abscissa of the center of gravity of the length l in the position shown is

$$\bar{x} = \frac{\frac{wax^2}{2}}{wal} = \frac{x^2}{2\,l}. \tag{1}$$

The velocity of the center of gravity is

$$\frac{d\bar{x}}{dt} = \frac{x}{l}\frac{dx}{dt} = \frac{x}{l}(-v) = -\frac{vx}{l}, \tag{2}$$

where v is the velocity of the stream.

Applying (4), § 181, to the system of particles,

$$F = \frac{d}{dt}\left(M\frac{d\bar{x}}{dt}\right). \tag{3}$$

In this case M is constant and equal to $\frac{wal}{g}$; therefore

$$F = \frac{wal}{g}\frac{d^2\bar{x}}{dt^2}. \tag{4}$$

Differentiating (2), $$\frac{d^2\bar{x}}{dt^2} = -\frac{v}{l}\frac{dx}{dt} = \frac{v^2}{l}, \tag{5}$$

and hence $$F = \frac{wav^2}{g}. \tag{6}$$

This result is usually obtained as follows: The force F is assumed constant, and then (2), § 183, gives

$$F(t - t_0) = mv - mv_0. \tag{7}$$

The time interval is taken as 1 sec., and therefore $m = \frac{w}{g}(av)$. Also the final horizontal velocity $v_0 = 0$. Hence

$$F = \frac{wav^2}{g}. \tag{8}$$

Substituting the numerical values from the problem in (6),

$$F = \frac{62.5}{32.2}\left(\frac{\pi}{4}\right)\left(\frac{1}{144}\right)(100)^2 = 106 \text{ lb.}$$

PROBLEMS

1. A body having an initial downward vertical velocity of 100 ft. per second falls under the action of gravity. Use the impulse-momentum equation to determine its velocity after 10 sec., air resistance being neglected. *Ans.* 422 ft. per second.

2. An electric car weighing 20 T. starts from rest and ascends a 3-per-cent grade. The driving force which the wheels exert upon the track is considered constant and equal to 3000 lb., and train resistance is 10 lb. per ton. Find the speed of the car after 10 sec. *Ans.* 12.88 ft. per second.

3. A body weighing 1 lb., initially at rest, is acted upon by an impulsive force which increases at a constant rate from 0 lb. to 3106 lb. and then decreases at the same rate to 0 lb. during an interval of 0.001 sec. Find the velocity of the body. *Ans.* 50 ft. per second.

4. A bullet weighing 1 oz. and having a speed of 1000 ft. per second is brought to rest by a constant force in 0.04 sec. Find the force. *Ans.* 48.5 lb.

5. A car weighing 50 T., moving with a speed of 10 ft. per second, strikes a second car, weighing 50 T., standing at rest. After the impact both cars move in the same direction with a speed of 5 ft. per

second. Find the impulse of the collision and the loss of energy during impact. How is the lost energy dissipated?

Ans. 15,528 lb.-sec., 38.82 ft.-tons.

6. During a storm 1 in. of rain fell in 2 hr. Find the pressure per square foot on a flat roof caused by the inelastic impact of the raindrops if the terminal velocity of the raindrops is 10 ft. per second. Also find the pressure per square mile. *Ans.* 0.000224 lb., 3.126 T.

7. A stream of water 2 in. in diameter strikes a curved stationary vane as shown in Fig. 322 with a speed of 100 ft. per second. Find the horizontal and vertical reactions, F and R, of the vane upon the stream. Also find F and R if the angle of the vane is 180°.

Ans. $F = 123.9$ lb., $R = 299.0$ lb.; $F = 845.7$ lb., $R = 0$.

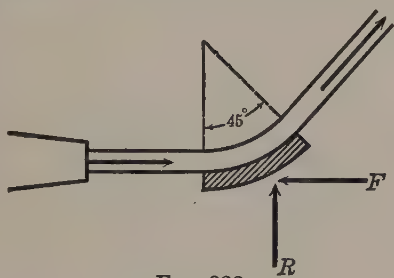

FIG. 322

8. Find the force which the water exerts upon the 180° vane of Problem 7 if the vane moves in the direction of the jet with a speed of 50 ft. per second. *Ans.* 211.4 lb.

9. A chain 10 ft. long, weighing 100 lb., is suspended vertically, with the lower end just touching a table. If the chain is released, find the pressure on the table when one half of the chain has coiled itself on the table. Find the pressure just as the upper end of the chain strikes the table. *Ans.* 150 lb., 300 lb.

10. After the chain of Problem 9 has fallen on the table a force is applied to one end of the chain to lift it with a uniform velocity of 8 ft. per second. Find the force when one half of the chain has been lifted. *Ans.* 69.9 lb.

11. The chain of Problem 9 is suspended vertically with its lower end 10 ft. from the table and then let go. Find the pressure on the table when one half of the chain has been deposited on it.

Ans. 350 lb.

12. A track scoop on the tender of a locomotive collects water at the rate of 80 cu. ft. per second when the train is running at a speed of 50 mi. per hour. Find the horizontal retarding force upon the tender. *Ans.* 1137 lb.

13. One thousand pounds of wheat is discharged vertically into a weighing truck in 4 sec. with a velocity which would be acquired in a free fall of 16 ft. Determine the total upward force of the rails on the wheels of the truck when one half of the wheat has been discharged. The truck weighs 500 lb. *Ans.* 1249 lb.

185. Moment of momentum of a rigid body rotating about a fixed axis. Let the z axis coincide with the fixed axis of rotation of a body whose angular velocity is ω. The linear momentum of any particle P of mass m at a distance r from the axis is $m\omega r$. The moment of momentum of this particle about the axis is $m\omega r^2$. The moment of momentum of the whole body is

$$\Sigma(m\omega r^2) = \omega\Sigma mr^2 = I\omega, \tag{1}$$

where I is the moment of inertia of the body with respect to the fixed axis.

The moment of momentum of a rigid body is frequently called the *angular momentum.*

The fundamental equation for rotation of a body about a fixed axis ((3), § 162),

$$N = I\alpha, \tag{2}$$

may be written

$$N = I\frac{d\omega}{dt}. \tag{3}$$

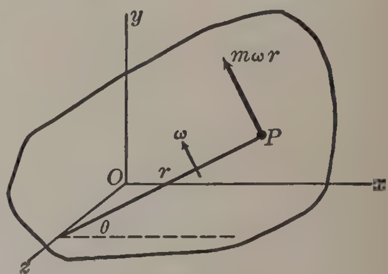

FIG. 323

Integrating with respect to t,

$$\int_{t_0}^{t} N\,dt = I(\omega - \omega_0,) \tag{4}$$

where ω and ω_0 are the angular velocities at times t and t_0 respectively.

If the forces constituting the couple N are very large and the time interval very small, the left-hand member of (4) is called an *impulsive couple.* The impulsive moment $\int_{t_0}^{t} N\,dt$ is sometimes designated by N_i.

186. Center of percussion. If a body is free to rotate about a fixed axis and if an impulsive force can be applied to the body so that there is no impulsive reaction at the axis, any point on the line of action of the impulsive force is called a center of percussion. As an example, consider a slender rod of length l and weight W free to rotate about a smooth pivot at its upper end. Let it be required to find the position of the center of percussion if a horizontal impulsive force, whose impulse is P_i, is applied to the rod at a distance d below the pivot. Let the

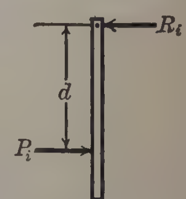

FIG. 324

rod be at rest before the action of the impulsive force, and let ω be the angular velocity imparted to the rod. The velocity of the center of gravity is $\frac{l\omega}{2}$, and the equation of motion of the center of gravity is, by (3), § 183,

$$P_i - R_i = \frac{W}{g}\frac{l}{2}(\omega - 0). \tag{1}$$

The equation of moment of momentum is, by (4), § 185,

$$P_i d = \frac{Wk^2}{g}(\omega - 0). \tag{2}$$

Eliminating P_i between (1) and (2),

$$R_i = \frac{W\omega}{g}\left(\frac{k^2}{d} - \frac{l}{2}\right). \tag{3}$$

The impulse R_i at the hinge vanishes when

$$\frac{k^2}{d} - \frac{l}{2} = 0,$$

that is, when
$$d = \frac{k^2}{\left(\frac{l}{2}\right)},$$

and hence the center of percussion coincides with the center of oscillation (§ 165).

The conditions that a body turning about a fixed axis may have a center of percussion are as follows:

1. The impulsive force must act perpendicular to the plane containing the axis of rotation and the instantaneous position of the center of gravity.

2. The axis of rotation must be a principal axis at some point in its length (§ 114).

3. The perpendicular distance between the line of action of the impulsive force and the axis of rotation must be equal to the distance of the center of oscillation from the axis.

The proof of these conditions is beyond the scope of this book.

EXAMPLES

1. A flywheel weighing 1000 lb. and having a radius of gyration of 3 ft. is running at 240 R.P.M. Find its angular velocity after 5 sec. if a moment $T = 30\,t^2$ lb.-ft. is applied to reduce the speed of the wheel.

Solution. Since the moment of the impulse is equal to the change in the angular momentum,

$$\int_0^5 30\,t^2\,dt = \frac{1000}{32.2}(3)^2(8\,\pi - \omega),$$

from which

$$\omega = 20.66 \text{ radians per second}$$
$$= 197 \text{ R.P.M.}$$

2. Find the position of the center of percussion of a sphere rotating about a fixed axis tangent to the sphere.

Solution. Since the center of percussion coincides with the center of oscillation, the distance from the axis to the center of percussion is (§ 186)

$$\frac{k^2}{h} = \frac{\frac{7}{5}\,r^2}{r} = \frac{7}{5}\,r.$$

PROBLEMS

1. A wheel and axle weighing 64.4 lb. has a radius of gyration of 2 ft. A string wound around the axle exerts a uniform moment of 5 lb.-ft. during an interval of 10 sec. Find the change in angular velocity if friction is neglected. *Ans.* 6.25 radians per second.

2. Show that the center of percussion of a uniform rod of length l rotating about a fixed axis through one end is $\frac{2}{3}\,l$. Is the result the same for a square plate whose side is l if the axis coincides with one edge?

3. A billiard ball rests on a table. Show that it must be struck at a point whose distance above the center is $\frac{2}{5}\,r$, in order that there may be no tendency for the point of contact to slip on the table.

4. A rod 6 ft. long, weighing 16.1 lb., is suspended at rest from a fixed pivot at one end. It is acted upon at the center by a constant horizontal impulsive force of 100,000 lb. during 0.001 sec. Find the angular velocity of the rod immediately after the action of the force. What fraction of the impulse P_i appears at the pivot?

Ans. 50 radians per second, 0.25 P_i.

5. Find the center of percussion of a pendulum consisting of a sphere 1 ft. in diameter weighing 20 lb. which is attached to the end of a thin rod 6 ft. long weighing 10 lb.

Ans. 6.044 ft. from the end of the rod.

6. A uniform rod AB 12 ft. long, weighing 30 lb., hangs vertically from a smooth horizontal axis at A. Find the points in the rod where a horizontal impulsive force must be applied so that the horizontal impulsive force at the axis may be one third of the applied impulsive force. *Ans.* 10.67 ft., or 5.33 ft. below A.

7. A slender rod 6 ft. long, weighing 32.2 lb., is suspended from a smooth horizontal axis 1 ft. from the upper end. Find its initial angular velocity if it is acted upon at the lower end by an impulsive force which would impart a velocity of 20 ft. per second to a mass of 4 slugs. *Ans.* $\omega = 57.1$ radians per second.

8. A rod weighing 64.4 lb., whose length is 6 ft., is free to turn about a fixed horizontal axis through one end. The rod is initially at rest with its center of gravity vertically above the axis. After the rod is given a slight displacement it falls through a right angle and the free end strikes a fixed inelastic peg. Find the impulse at the peg and at the axis. *Ans.* 16.05 lb.-sec., 8.02 lb.-sec.

9. A uniform rod, of weight W and length l, is balanced about a horizontal axis through its center of gravity. Find the angular velocity imparted to it by an impulsive force of impulse P_i applied normal to the rod at one end. *Ans.* $\dfrac{6\,gP_i}{Wl}$.

187. The moment of momentum of a body having plane motion. Let $(\bar{x}, \bar{y})$ be the coördinates of the center of gravity of a body having plane motion, and let (x, y) be the coördinates of any particle of mass m of the body, both sets of coördinates being referred to axes fixed in the plane of motion. Further, let (x', y') be the coördinates of the particle referred to axes which remain parallel to the fixed axes and which have their origin at $G(\bar{x}, \bar{y})$, the center of gravity of the body. Then, for any particle of the body,

$$x = \bar{x} + x', \tag{1}$$

and

$$y = \bar{y} + y'. \tag{2}$$

The moment of momentum of any particle with respect to an axis perpendicular to the plane of motion passing through the origin O of the fixed axes is, by § 182,

$$m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right). \tag{3}$$

Substituting the values of x, y, $\dfrac{dx}{dt}$, and $\dfrac{dy}{dt}$ from (1) and (2)

in the expression (3), the moment of momentum of the body with respect to the fixed axis through O is

$$\Sigma m\left[(\bar{x}+x')\left(\frac{d\bar{y}}{dt}+\frac{dy'}{dt}\right)-(\bar{y}+y')\left(\frac{d\bar{x}}{dt}+\frac{dx'}{dt}\right)\right], \tag{4}$$

or
$$\Sigma m\left(\bar{x}\,\frac{d\bar{y}}{dt}-\bar{y}\,\frac{d\bar{x}}{dt}\right)+\Sigma m\left(x'\,\frac{dy'}{dt}-y'\,\frac{dx'}{dt}\right)$$
$$+\Sigma m\left(\bar{x}\,\frac{dy'}{dt}+x'\,\frac{d\bar{y}}{dt}-\bar{y}\,\frac{dx'}{dt}-y'\,\frac{d\bar{x}}{dt}\right). \tag{5}$$

Multiplying (1) by m and extending the summations throughout the body,

$$\Sigma mx=\Sigma m(\bar{x}+x')=\Sigma m\bar{x}+\Sigma mx'.$$

But $$\Sigma mx=M\bar{x},$$

and also $$\Sigma m\bar{x}=M\bar{x}.$$

$$\left.\begin{aligned}&\text{Hence}\quad \Sigma mx'=0,\ \text{ and therefore }\ \Sigma\left(m\,\frac{dx'}{dt}\right)=0.\\ &\text{Likewise}\quad \Sigma my'=0,\ \text{ and }\ \Sigma\left(m\,\frac{dy'}{dt}\right)=0.\end{aligned}\right\} \tag{6}$$

Substituting the values obtained in (6) in the expression (5), the moment of momentum of the body with respect to the fixed axis through O becomes

$$M\left(\bar{x}\,\frac{d\bar{y}}{dt}-\bar{y}\,\frac{d\bar{x}}{dt}\right)+\Sigma m\left(x'\,\frac{dy'}{dt}-y'\,\frac{dx'}{dt}\right). \tag{7}$$

Let Mv be the momentum vector of a particle of mass M moving with the center of gravity of the body, and let d be the perpendicular distance from the axis O to the momentum vector. Then, from Fig. 325 and Varignon's theorem, the moment of momentum of the particle of mass M about the axis through O is

$$Mvd=M\left(\bar{x}\,\frac{d\bar{y}}{dt}-\bar{y}\,\frac{d\bar{x}}{dt}\right). \tag{8}$$

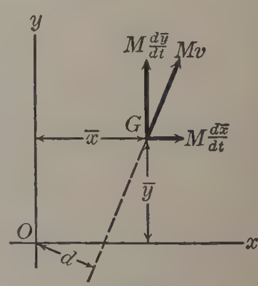

FIG. 325

Also, by § 185, the moment of momentum of the body about an axis through G perpendicular to the plane of motion is

$$\Sigma\left[m\left(x'\,\frac{dy'}{dt}-y'\,\frac{dx'}{dt}\right)\right]=I_G\omega=Mk^2\omega. \tag{9}$$

Substituting the values obtained in (8) and (9) in the expression (7), the moment of momentum of the body about an axis through O perpendicular to the plane of motion is

$$I_G\omega + Mvd. \tag{10}$$

Therefore *the moment of momentum of a body of mass M with respect to a fixed axis perpendicular to its plane of motion is equal to the moment of momentum with respect to the same axis of a particle of mass M moving with the center of gravity, plus the moment of momentum of the body with respect to a parallel axis through the center of gravity.*

188. Impulsive forces in plane motion. A body having plane motion is acted upon by impulsive forces whose component impulses are $\Sigma\int_{t_0}^{t} X\,dt$ and $\Sigma\int_{t_0}^{t} Y\,dt$; it is required to find the changes in its motion.

Let (u_0, v_0) and (u, v) be the component velocities of the center of gravity of the body just before and just after the action of the impulsive forces. Then, from equation (3), § 183, the change in the motion of the center of gravity is given by

$$\Sigma\int_{t_0}^{t} X\,dt = M(u-u_0) \quad \text{and} \quad \Sigma\int_{t_0}^{t} Y\,dt = M(v-v_0). \tag{1}$$

The equation of angular momentum about an axis perpendicular to the plane of motion through the center of gravity, referred to a set of translating axes whose origin is at the center of gravity, is

$$N_i = I(\omega - \omega_0), \tag{2}$$

where ω_0 and ω are the angular velocities of the body just before and just after the action of the impulsive forces, the moment of whose impulses about the axis through the center of gravity is N_i. Equation (2) may be derived from the fundamental equations of motion or more simply from equation (2), case (*b*), § 179, in a manner similar to the derivation of equation (4) from equation (3) of § 185.

189. Conservation of momentum. Since by § 181 the time rate of change of the linear momentum of a body in any direction is equal to the sum of the external forces in that direction, it follows that if a direction can be found such that the sum of the external forces in that direction is zero, the linear

momentum of the body in that direction is constant. This principle is known as the principle of the conservation of linear momentum.

Also, since by § 182 the time rate of change of the angular momentum of a body about a fixed axis is equal to the sum of the moments of the external forces about that axis, it follows that if an axis can be found such that the sum of the moments of the external forces about that axis is zero, the angular momentum about that axis is constant. This principle is known as the principle of the conservation of angular momentum. Similar reasoning may be employed to show that these principles are valid for impulsive forces.

EXAMPLES

1. A wooden block weighing 2 lb. is moving on a smooth horizontal plane with a velocity of 10 ft. per second when it is struck by a bullet weighing 1 oz. moving horizontally with a velocity of 1000 ft. per second in the same direction. Find the common velocity of the block and bullet just after impact.

Solution. The system comprising the bullet and the block is free from external force in the direction of the motion during the impact. Hence the momentum of the system before impact is equal to its momentum after impact. Therefore, if the factor $\frac{1}{g}$ is omitted,

$$\tfrac{1}{16}(1000) + 2(10) = (2 + \tfrac{1}{16})v,$$

from which $$v = 40 \text{ ft. per second.}$$

2. A circular disk of radius 2 ft. and weighing 20 lb. rolls on a horizontal plane with a speed of 12 ft. per second. Find its angular momentum with respect to a polar axis through the instantaneous center.

Solution. First method. By § 187, the moment of momentum of the disk with respect to a polar axis through the instantaneous center is the moment of momentum with respect to that axis of a particle of mass $\frac{20}{g}$ at the center of gravity moving with a speed of 12 ft. per second plus the moment of momentum $I\omega$ of the disk with respect to a parallel axis through its center of gravity.

Hence the required moment of momentum is

$$\frac{20}{g}(12)(2) + \frac{1}{2}\left(\frac{20}{g}\right)(2)^2\left(\frac{12}{2}\right) = 22.4 \text{ lb.-ft.-sec.}$$

Second method. The moment of momentum equals $I\omega$, where I is the moment of inertia with respect to the instantaneous axis, or

$$\frac{3}{2}\left(\frac{20}{g}\right)(2)^2\left(\frac{12}{2}\right) = 22.4 \text{ lb.-ft.-sec.}$$

It is to be noticed that for a body having plane motion it is valid to express the moment of momentum of a body about an axis as the product of the moment of inertia about that axis and the angular velocity of the body in the three following cases only:

(*a*) When the axis passes through the center of gravity.
(*b*) When the axis is fixed in the body and in space.
(*c*) When the axis coincides with the instantaneous axis.

In other cases the moment of momentum is

$$Mvd + I_G\omega,$$

where I_G is the moment of inertia about a parallel axis through the center of gravity.

3. A uniform circular disk, weighing 60 lb. and having a radius of 2 ft., slides upon a horizontal plane with a linear velocity of 20 ft. per second and an angular velocity of 20 radians per second. Find its angular velocity if it is suddenly constrained to rotate about an axis through its circumference perpendicular to its plane.

Solution. Taking the axes of reference as shown in Fig. 326, the polar coördinates of any axis of attachment P are $(2, \theta)$.

Since the impulse at P caused by the sudden fixation of the axis has no moment with respect to P, the moment of momentum with respect to the axis at P will not be changed by the impact. The moment of momentum with respect to the axis P before fixation is, by § 187,

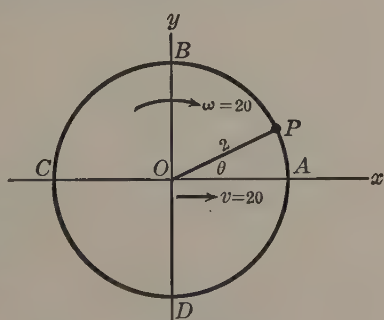

Fig. 326

$$Mvd + I_G\omega,$$

which becomes $\left[\frac{60}{g} \times 20 \times 2 \sin\theta - \frac{1}{2} \times \frac{60}{g} \times (2)^2 \times 20\right]$ lb.-ft.-sec.

The moment of momentum after impact is $I\omega$, where I is the moment of inertia with respect to the axis at P, or $\frac{3}{2} \times \frac{60}{g} \times (2)^2\,\omega$.

By the principle of conservation of angular momentum,

$$\frac{3}{2} \times \frac{60}{g} \times (2)^2\,\omega = \frac{60}{g} \times 20 \times 2 \sin\theta - \frac{1}{2} \times \frac{60}{g} \times (2)^2 \times 20,$$

from which $$\omega = \tfrac{20}{3}(\sin\theta - 1).$$

At the point A, $\theta = 0$ and $\omega = -\frac{20}{3}$ radians per second.

At the point B, $\theta = 90°$ and $\omega = 0$ radians per second.

At the point C, $\theta = 180°$ and $\omega = -\frac{20}{3}$ radians per second.

At the point D, $\theta = 270°$ and $\omega = -\frac{40}{3}$ radians per second.

The disk turns clockwise after fixation about all axes through the circumference except at D. At D the disk is reduced to rest.

4. A man stands upon a table which is free to revolve about a vertical axis through his center of gravity. Show that if the man is given an angular velocity while his arms are extended, his angular velocity will be increased if he permits his arms to fall to his sides.

Solution. If the small moment of friction and air resistance is neglected, the angular momentum of the man and the table remains constant. Since the man decreases the moment of inertia of the system when he permits his arms to fall, the angular velocity must increase in order to keep the angular momentum $I\omega$ constant.

5. A uniform rod of mass M and length l, resting on a smooth horizontal plane, is acted upon by an impulsive force at right angles to the rod and at a distance d from its center. Find the *axis of spontaneous rotation*, that is, the axis about which the rod begins to rotate.

Solution. Let v be the velocity of the center of gravity and ω the angular velocity after the action of the impulsive force whose impulse is P_i. Also let q be the distance from the center of gravity to the axis of spontaneous rotation. Then

$$q\omega = v,$$

and

$$P_i = Mq\omega$$

Also

$$P_i(d + q) = (\tfrac{1}{12}Ml^2 + Mq^2)\omega.$$

Solving for ω and q,

$$\omega = \frac{12\,P_i d}{Ml^2} \quad \text{and} \quad q = \frac{l^2}{12\,d}.$$

PROBLEMS

1. A car weighing 40 T., moving 2 ft. per second, collides and couples with a car weighing 30 T., moving in the opposite direction with a velocity of 1 ft. per second. Find the velocity after impact.
Ans. 0.71 ft. per second.

2. A rifle weighing 6 lb. fires a bullet weighing 1 oz. with a muzzle velocity of 2000 ft. per second. Find the initial velocity of recoil of the rifle if it is free to move. Neglect the mass of the powder and the gases.
Ans. 20.83 ft. per second.

3. Two particles, weighing 16 lb. each, are connected by an inextensible string 10 ft. long. Initially the particles are together and the first particle is projected upward with a speed of 40 ft. per second. Find the velocity of the particles just after the string becomes taut. *Ans.* 15.46 ft. per second.

4. A cat is held with its feet upward and let go. Explain how it turns its body in order to fall on its feet.

5. A uniform rod 6 ft. long rotates on a smooth horizontal plane with an angular velocity of 60 R.P.M. Find the angular velocity if one end is suddenly fixed. *Ans.* 15 R.P.M.

6. Find the position of fixation of an axis in the rod of Problem 5 so that the angular velocity of the rod may be reduced to 30 R.P.M. *Ans.* 1.732 ft. from the center.

7. A circular disk of radius r rests on a smooth horizontal table. Where should it be acted upon by a horizontal impulsive force so that it may begin to rotate about an axis perpendicular to its plane through a point in its circumference?

Ans. In the plane of the disk perpendicular to a radius at a distance of $\frac{r}{2}$ from the center.

8. A compound pendulum in the form of an elliptical lamina, whose major and minor axes are $2a$ and $2b$ respectively, oscillates about a polar axis through an extremity of the major axis. Where should it be struck at each oscillation so that there shall be no impulsive reaction at the axis? *Ans.* $\frac{5a^2 + b^2}{4a}$ from the axis.

9. A uniform plate, in the form of a square whose side is a, revolves about a polar axis through its center with an angular velocity of 20 R.P.M. Find the angular velocity if the plate suddenly begins to revolve about a parallel axis through one of its corners. *Ans.* 5 R.P.M.

10. A thin hoop of radius r revolves in its plane about a polar axis through its center with an angular velocity of 30 R.P.M. Find the angular velocity if a parallel axis through a point in its circumference is suddenly fixed. *Ans.* 15 R.P.M.

11. A gear running clockwise at 60 R.P.M. suddenly meshes with another gear exactly like it running counterclockwise at 40 R.P.M. Find the angular velocity after meshing. *Ans.* 50 R.P.M.

12. A flat circular plate rotates in its plane with an angular velocity ω about a polar axis which pierces the plate at a point midway between its center and a point on its circumference. Find

the angular velocity of the plate if the extremity of a diameter through the point A becomes suddenly fixed to a parallel axis.

Ans. $\frac{2}{3}\,\omega$ or 0.

13. A solid hemisphere having a radius of 3 ft. rotates about a horizontal diameter with an angular velocity of 52 R.P.M. Find the angular velocity if the extremity of the radius perpendicular to the base is suddenly fixed to an axis parallel to the first axis of rotation. *Ans.* 2 R.P.M.

14. A homogeneous circular cylinder of radius r which rolls on a rough horizontal plane has a polar axis suddenly fixed in its circumference at a distance h above the plane. Find the angular velocity which the cylinder must have in order that it shall just come to rest with its center above the new axis of rotation.

Ans. $\omega = \dfrac{2\sqrt{3\,gh}}{3\,r - 2\,h}.$

15. A turntable of radius 4 ft. and weighing 100 lb. is in the form of a uniform circular disk free to rotate about a vertical axis through its center. A man weighing 160 lb. walks completely around the outer edge of the turntable. Find the angle turned through by the turntable. *Ans.* 274.3°.

190. Elastic impact. If two bodies collide without rebound, the impact is said to be inelastic. If rebound occurs, the impact is said to be elastic. During the first stage of a central elastic impact both bodies are being compressed while their velocities are being altered. At the end of the first stage of impact the bodies have received their maximum compression and they are moving with a common velocity. During the second stage of the impact the elastic restoring forces act upon the bodies to give them individual velocities which depend upon their masses. It is found by experiment that when two smooth elastic spheres moving in the same straight line collide, the relative velocity of the spheres just after separation bears a definite ratio to their relative velocity just before contact. This ratio, designated by e, is called the coefficient of restitution and depends upon the material of the spheres. For balls of soft clay it is nearly zero and for balls of hardened steel it is nearly unity. Since e is never quite unity a loss of energy always accompanies an impact.

191. The direct impact of a body on a fixed surface. The impact of a body is said to be direct if it moves along a line normal to the surface of contact at the instant of impact. Let m be the

mass of a body, and let v be its velocity before impact and v' its velocity after impact on a fixed surface. Then, by the definition of the coefficient of restitution,

$$-ev = v'.$$

The minus sign is placed before e, since the relative velocities of separation and approach are always opposite to each other in sense.

192. The direct impact of two moving bodies. Let M_1 and M_2 be the masses of two bodies which collide with direct central impact. Also let

v_1 = the velocity of mass M_1 before impact,

v_2 = the velocity of mass M_2 before impact,

v'_1 = the velocity of mass M_1 after impact,

v'_2 = the velocity of mass M_2 after impact,

and e = the coefficient of restitution.

Then, since the external forces acting upon the bodies may be neglected in comparison with the impulsive forces caused by the collision, the total momentum of the system remains unchanged by the impact (§§ 183 and 189). Hence

$$M_1v_1 + M_2v_2 = M_1v'_1 + M_2v'_2. \tag{1}$$

Also, since the relative velocity of separation of the bodies is equal to the product of $-e$ and the relative velocity of approach,

$$-e(v_1 - v_2) = (v'_1 - v'_2). \tag{2}$$

Either direction on the line of impact may be chosen as positive, but when the choice is once made, the signs should be properly affixed if they are known; if they are not known they should be written as they appear in the formulas. The general case of the impact of two bodies is beyond the scope of this book.

EXAMPLES

1. A ball falls a vertical distance h from rest and strikes a fixed horizontal plate. Find the height of rebound.

Solution. The velocities of the ball just before and just after impact are $\sqrt{2\,gh}$ and $-e\sqrt{2\,gh}$ respectively. If h' is the height of rebound,

$$-e\sqrt{2\,gh} = \sqrt{2\,gh'},$$

from which $$h' = e^2h.$$

2. A smooth sphere having a velocity v strikes a fixed smooth plate at an angle θ with the normal to the plate. Find the velocity of the sphere after impact and the angle which the velocity makes with the normal if the coefficient of restitution is e.

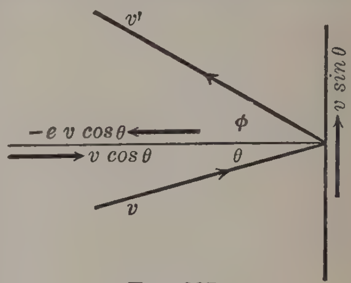

FIG. 327

Solution. Let the velocity v be resolved into two components, $v \cos \theta$ and $v \sin \theta$, along the normal and perpendicular to it respectively. Since the plate is smooth there is no change of momentum parallel to it, and hence the component $v \sin \theta$ remains unchanged by the impact. The component velocity normal to the plate after impact is $-ev \cos \theta$. Hence the total velocity after impact is

$$v' = \sqrt{(v \sin \theta)^2 + (-ev \cos \theta)^2} = v\sqrt{\sin^2 \theta + e^2 \cos^2 \theta}.$$

The angle ϕ which the path of the sphere makes with the normal as it leaves the plate is given by

$$\tan \phi = \frac{v \sin \theta}{ev \cos \theta} = \frac{\tan \theta}{e}.$$

3. A car weighing 40 T. is moving east on a straight track with a speed of 5 ft. per second, when it strikes a second car weighing 30 T. moving west at a speed of 6 ft. per second. Find the velocities after impact if the coefficient of restitution is 0.2.

Solution. Let the direction of motion of the 40-ton car be positive. Then $v_1 = 5$ ft. per second and $v_2 = -6$ ft. per second. Hence, by § 192,

$$(40)(5) + (30)(-6) = 40\,v'_1 + 30\,v'_2.$$

The equation of relative velocity is

$$-0.2\,[5 - (-6)] = v'_1 - v'_2.$$

Solving for v'_1 and v'_2,

$$v'_1 = -0.66 \text{ ft. per second}$$

and

$$v'_2 = 1.54 \text{ ft. per second.}$$

The minus sign before v'_1 indicates that the first car moves in the westward direction after impact.

4. A target weighing 2 lb., consisting of a flat circular plate 1 ft. in diameter, is free to rotate about a horizontal axis tangent to its circumference. Find the initial angular velocity of the target if it is struck normally at the center by a bullet weighing 1 oz. which has a velocity of 1600 ft. per second. The coefficient of restitution is 0.2.

Solution. Let M be the mass of the target and m the mass of the bullet. Since the system consisting of the target and the bullet is not acted upon

by any external couple during impact, the total angular momentum with respect to the axis A must remain constant (§ 189). The moment of momentum of the bullet about the axis A before impact is mv_1r, where v_1 is the velocity of the bullet and r is the radius of the target. The angular momentum of the target before impact is zero. Hence

$$mv_1r = mv'_1r + \tfrac{5}{4} Mr^2\omega'_2,$$

where $\frac{5}{4} Mr^2$ is the moment of inertia of the target with respect to the axis at A, and v'_1 and ω'_2 are the linear velocity of the bullet and the angular velocity of the target, respectively, after impact.

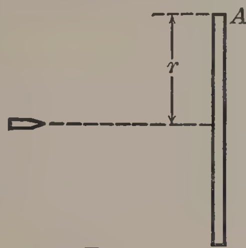

FIG. 328

Also $$-ev_1 = v'_1 - \omega'_2r,$$

where ω'_2r is the velocity after impact of the point of the target struck by the bullet. Substituting the numerical values in the equations and solving,

$$\omega'_2 = 93.7 \text{ radians per second}$$

and $$v'_1 = -273.2 \text{ ft. per second.}$$

The minus sign indicates that the bullet reverses its direction.

PROBLEMS

1. An elastic ball strikes a smooth fixed plate at an angle of 30° with the normal. The angle of rebound is 45°. Find the coefficient of restitution. *Ans.* $e = 0.577$.

2. A ball which falls from rest a distance of 64 ft. onto a floor is found to reach a height of 1 ft. on the third rebound. Find the coefficient of restitution. *Ans.* $e = 0.5$.

3. A sphere of mass m_1 strikes a stationary sphere of mass m_2, the impact being direct. After the impact the velocity of the mass m_1 is in the same direction and equal to one half of the initial velocity. Find the ratio of the mass m_1 to m_2. *Ans.* $\dfrac{m_1}{m_2} = 1 + 2e$.

4. Two elastic spheres having the same velocities in opposite directions collide with each other. After the impact the first sphere has a velocity of one fourth of its initial velocity but in the opposite direction. Find the ratio of the masses of the spheres. *Ans.* $\dfrac{3+8e}{5}$.

5. Two small spheres, of masses M and $\dfrac{M}{2}$ respectively, are suspended by light rods of length l. If the sphere of mass M is drawn aside through an angle θ and permitted to fall, find the initial velocity of the sphere of mass $\dfrac{M}{2}$. The sphere of mass $\dfrac{M}{2}$ is at rest before impact.

Ans. $\frac{2}{3}(1 + e)\sqrt{2gl(1 - \cos\theta)}$.

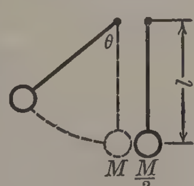

FIG. 329

6. A rod 3 ft. long, weighing 10 lb., is pivoted at the upper end to swing in a vertical plane. The rod is displaced through an angle of 60° with the vertical and let go. When the rod reaches the vertical position its lower end strikes a small block weighing 1 oz. which is at rest on a horizontal plane ($\mu = 0.5$). Find the distance the block moves on the plane if e is 0.8. *Ans.* 14.05 ft.

7. A ball A weighing 10 lb. is supported by a light rod 9 ft. long pivoted at D, Fig. 330. The ball A falls through 90° and impinges upon ball B, which in turn impinges upon ball C. Find the vertical distance d through which the ball C moves if $e = 0.8$ and if the weights of balls B and C are 8 lb. and 4 lb. respectively. *Ans.* 12.96 ft.

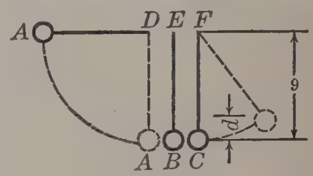

FIG. 330

8. A uniform rod, of weight W_1 and length l, is pivoted at its upper end. The rod falls from a horizontal position and strikes a block of weight W_2, which is at rest on a smooth horizontal plane. Find the velocity of the block immediately after the impact.

$$Ans.\ \frac{W_1\sqrt{3\,gl}\,(1+e)}{W_1 + 3\,W_2}.$$

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